

UMD BANACH SPACES AND SQUARE FUNCTIONS ASSOCIATED WITH HEAT SEMIGROUPS FOR SCHRÖDINGER AND LAGUERRE OPERATORS

J.J. BETANCOR, A.J. CASTRO, J.C. FARIÑA, AND L. RODRÍGUEZ-MESA

ABSTRACT. In this paper we define square functions (also called Littlewood-Paley-Stein functions) associated with heat semigroups for Schrödinger and Laguerre operators acting on functions which take values in UMD Banach spaces. We extend classical (scalar) L^p -boundedness properties for the square functions to our Banach valued setting by using γ -radonifying operators. We also prove that these L^p -boundedness properties of the square functions actually characterize the Banach spaces having the UMD property.

1. INTRODUCTION

Suppose that (Ω, μ) is a measure space and $\{T_t\}_{t>0}$ is an analytic semigroup on $L^p(\Omega, \mu)$, where $1 \leq p \leq \infty$. If $k \in \mathbb{N}$, the k -th vertical square function $g^k(\{T_t\}_{t>0})(f)$ of $f \in L^p(\Omega, \mu)$ is defined by

$$g^k(\{T_t\}_{t>0})(f)(x) = \left(\int_0^\infty |t^k \partial_t^k T_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

The L^p -boundedness properties of g^k -square functions are very useful in order to describe the behavior in L^p -spaces of multipliers associated to the infinitesimal generator of the semigroup $\{T_t\}_{t>0}$ (see [26], [29] and [33]).

It is well-known ([29, p. 120]) that if $\{T_t\}_{t>0}$ is the classical heat or Poisson semigroup then, for every $1 < p < \infty$,

$$(1) \quad \|g^k(\{T_t\}_{t>0})(f)\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

This property can be extended to other semigroups of operators (see [25], [29], [33], [39], amongst others).

In the sequel we denote as usual by $\{W_t\}_{t>0}$ and $\{P_t\}_{t>0}$ the classical heat and Poisson semigroup on \mathbb{R}^n , respectively. We have that, for every $t > 0$ and $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$,

$$W_t(f)(x) = \int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/(4t)}}{(4\pi t)^{n/2}} f(y) dy, \quad x \in \mathbb{R}^n,$$

and

$$P_t(f)(x) = c_n \int_{\mathbb{R}^n} \frac{t}{(t^2 + |x-y|^2)^{(n+1)/2}} f(y) dy, \quad x \in \mathbb{R}^n,$$

being $c_n = \pi^{-(n+1)/2} \Gamma((n+1)/2)$.

If $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a measurable function on \mathbb{R}^n , we define $\psi_t(x) = t^{-n} \psi(x/t)$, $x \in \mathbb{R}^n$ and $t > 0$. Then, it is clear that, for every $t > 0$, $W_t(f) = G_{\sqrt{t}} * f$ and $P_t(f) = P_t * f$, where

Date: Friday 21st September, 2012.

2010 *Mathematics Subject Classification.* 46E40, 46B20.

Key words and phrases. γ -radonifying operators, UMD Banach spaces, Schrödinger, Hermite and Laguerre operators, Littlewood-Paley g -functions, heat semigroup.

$G(x) = \frac{e^{-|x|^2/4}}{(4\pi)^{n/2}}$, $x \in \mathbb{R}^n$, and $P(x) = \frac{c_n}{(1+|x|^2)^{(n+1)/2}}$, $x \in \mathbb{R}^n$. We can also write, for every $k \in \mathbb{N}$,

$$g^k(\{W_t\}_{t>0})(f)(x) = \sqrt{2} \|\varphi_{\sqrt{t}}^k * f(x)\|_{L^2((0,\infty), \frac{dt}{t})},$$

where $\varphi^k(x) = \left(\partial_t^k G_{\sqrt{t}}(x)\right)_{|t=1}$, $x \in \mathbb{R}^n$, and

$$g^k(\{P_t\}_{t>0})(f)(x) = \|\phi_t^k * f\|_{L^2((0,\infty), \frac{dt}{t})},$$

where $\phi^k(x) = \left(\partial_t^k P_t(x)\right)_{|t=1}$, $x \in \mathbb{R}^n$.

If ψ is good enough the continuous ψ -wavelet transform $\mathcal{W}_\psi(f)$ of $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, is defined by

$$\mathcal{W}_\psi(f)(x, t) = (\psi_t * f)(x), \quad x \in \mathbb{R}^n \text{ and } t > 0.$$

In [8] (see also [14]) the authors gave conditions on the function ψ so that the equivalence

$$(2) \quad \|\mathcal{W}_\psi(f)\|_{L^p(\mathbb{R}^n, L^2((0,\infty), \frac{dt}{t}))} \sim \|f\|_{L^p(\mathbb{R}^n)},$$

holds for every $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Note that (2) can be seen as an extension of (1) for the classical heat and Poisson semigroups.

In the last years several authors ([17], [19], [20], [22], [23], [25] and [39]) have dealt with square functions acting on functions which take values in a Banach space. Suppose that \mathbb{B} is a Banach space and $f : \Omega \rightarrow \mathbb{B}$ is a μ -strongly measurable function. The first (and maybe the more natural) definition of $g_{\mathbb{B}}^k(\{T_t\}_{t>0})(f)$ is the following:

$$g_{\mathbb{B}}^k(\{T_t\}_{t>0})(f)(x) = \left(\int_0^\infty \|t^k \partial_t^k T_t(f)(x)\|_{\mathbb{B}}^2 \frac{dt}{t} \right)^{1/2}.$$

This $g_{\mathbb{B}}^k$ -square function was studied for the classical Poisson semigroup on the torus by Xu ([39]); for the Poisson semigroup defined by the Ornstein-Uhlenbeck semigroup by Harboure, Torrea and Viviani ([17]); for subordinated Poisson semigroups of diffusion semigroups (in the sense of Stein [29]) by Martínez, Torrea and Xu ([25]); and for Poisson semigroups associated with Schrödinger operators by Torrea and Zhang ([34]). From the results in [25] and [39] we can deduce the following.

Theorem. *Let \mathbb{B} be a Banach space and $1 < p < \infty$. Then, the following assertions are equivalent.*

- (i) \mathbb{B} is isomorphic to a Hilbert space.
- (ii) For every $f \in L^p(\mathbb{R}^n, \mathbb{B})$,

$$\|g_{\mathbb{B}}^1(\{P_t\}_{t>0})(f)\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}.$$

Other authors ([19], [20], [22] and [23]) have extended the definition of the g -square functions to a Banach valued setting by different points of view. As one of their goals, they wanted to extend the equivalence in (1) to Banach spaces which are not isomorphic to Hilbert spaces. Hytönen [19] extended (1) to a UMD Banach space setting by using Banach-valued stochastic integration. On the other hand, Kaiser and Weis [23] generalized (2) to functions taking values in UMD Banach spaces by using γ -radonifying operators. These two approaches are closely connected (see, for instance, [36] and [37]). In this paper we use γ -radonifying operators to study g -square functions associated with the heat semigroups for Schrödinger and Laguerre operators in UMD Banach spaces.

The main properties of UMD Banach spaces can be encountered in [6], [7] and [27].

Suppose that H is a separable Hilbert space and \mathbb{B} is a real Banach space. We take a sequence $(\gamma_k)_{k \in \mathbb{N}}$ of independent standard Gaussians. We say that an operator T bounded from H into \mathbb{B} , shortly $T \in L(H, \mathbb{B})$, is γ -radonifying, written $T \in \gamma(H, \mathbb{B})$, when

$$\|T\|_{\gamma(H, \mathbb{B})} = \left(\mathbb{E} \left\| \sum_{k=1}^{\infty} \gamma_k T(h_k) \right\|_{\mathbb{B}}^2 \right)^{1/2} < \infty,$$

where $\{h_k\}_{k \in \mathbb{N}}$ is an orthonormal basis in H . If \mathbb{B} is a Banach space not containing a copy of c_0 (that is the case of UMD spaces), then

$$(3) \quad \|T\|_{\gamma(H, \mathbb{B})} = \sup \left(\mathbb{E} \left\| \sum_{k=1}^{\infty} \gamma_k T(h_k) \right\|_{\mathbb{B}}^2 \right)^{1/2},$$

where the supremum is taken over all the finite families $\{h_k\}$ of orthonormal functions in H ([35, Theorem 5.9]). In the sequel by H we denote the space $L^2((0, \infty), dt/t)$.

If $f : (0, \infty) \rightarrow \mathbb{B}$ is a strongly μ -measurable function such that, for every $L \in \mathbb{B}^*$, $L \circ f \in H$, then there exists $T_f \in L(H, \mathbb{B})$ such that

$$\langle L, T_f(h) \rangle = \int_0^\infty \langle L, f(t) \rangle_{\mathbb{B}^*, \mathbb{B}} h(t) \frac{dt}{t}, \quad h \in H \text{ and } L \in \mathbb{B}^*.$$

We say that $f \in \gamma((0, \infty), dt/t, \mathbb{B})$ provided that $T_f \in \gamma(H, \mathbb{B})$. We identify f with T_f . If \mathbb{B} does not contain a copy of c_0 then $\gamma((0, \infty), dt/t, \mathbb{B})$ is a dense subspace of $\gamma(H, \mathbb{B})$ ([23, Remark 2.16]). In the sequel we assume that \mathbb{B} is UMD. Then, \mathbb{B} does not contain a copy of c_0 .

In [23, Theorem 4.2] Kaiser and Weis gave conditions over the function ψ in order to the wavelet transform \mathcal{W}_ψ satisfies the following equivalence:

$$(4) \quad \|\mathcal{W}_\psi(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \sim \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})},$$

for every $f \in L^p(\mathbb{R}^n, \mathbb{B})$ and $1 < p < \infty$. Note that, since $\gamma(H, \mathbb{C}) = H$, (4) reduces to (2) when $\mathbb{B} = \mathbb{C}$. Then, (4) can be seen as an extension of (1) when we consider the classical heat or Poisson semigroups and functions taking values in a UMD Banach space.

In this paper we extend the equivalence (1) to a UMD-Banach valued setting for the heat semigroup defined by Schrödinger operator in \mathbb{R}^n , $n \geq 3$, the Hermite operator on \mathbb{R}^n , $n \geq 1$, and the Laguerre operator on $(0, \infty)$. Then, we prove that these new equivalences allow us to characterize the UMD Banach spaces.

The Schrödinger operator \mathcal{L} is defined by $\mathcal{L} = -\Delta + V$ in \mathbb{R}^n , $n \geq 3$, where Δ is the Euclidean Laplacian in \mathbb{R}^n and V is a nonnegative measurable function in \mathbb{R}^n . Here we assume that $V \in RH_s(\mathbb{R}^n)$, that is, V satisfies the following s -reverse Hölder's inequality: there exists $C > 0$ such that, for every ball B in \mathbb{R}^n ,

$$(5) \quad \left(\int_B V(x)^s dx \right)^{1/s} \leq C \int_B V(x) dx,$$

where $s > n/2$. If $E_{\mathcal{L}}$ represents the spectral measure associated with the operator \mathcal{L} , the heat semigroup of operators generated by $-\mathcal{L}$ is denoted by $\{W_t^{\mathcal{L}}\}_{t>0}$, where

$$W_t^{\mathcal{L}}(f) = \int_{[0, \infty)} e^{-\lambda t} E_{\mathcal{L}}(d\lambda) f, \quad f \in L^2(\mathbb{R}^n).$$

We can write, for every $f \in L^2(\mathbb{R}^n)$,

$$(6) \quad W_t^{\mathcal{L}}(f)(x) = \int_{\mathbb{R}^n} W_t^{\mathcal{L}}(x, y) f(y) dy, \quad x \in \mathbb{R}^n \text{ and } t > 0.$$

The main properties of the kernel function $W_t^{\mathcal{L}}(x, y)$, $t > 0$, $x, y \in \mathbb{R}^n$, can be encountered in [9] and [28]. Also, for every $t > 0$, the operator $W_t^{\mathcal{L}}$ defined in (6) is bounded from $L^p(\mathbb{R}^n)$ into itself, $1 \leq p \leq \infty$. Thus, $\{W_t^{\mathcal{L}}\}_{t>0}$ is a positive semigroup of contractions in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

The Hermite (also called harmonic oscillator) operator $\mathcal{H} = -\Delta + |x|^2$ is a special case of the Schrödinger operator. Here we consider \mathcal{H} on \mathbb{R}^n , with $n \geq 1$. We define, for every $k \in \mathbb{N}$, the k -th Hermite function \mathfrak{h}_k by

$$\mathfrak{h}_k(x) = (\sqrt{\pi} 2^k k!)^{-1/2} e^{-x^2/2} H_k(z), \quad x \in \mathbb{R},$$

where by H_k we denote the k -th Hermite polynomial ([31, pp. 105–106]). If $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ the k -th Hermite function \mathfrak{h}_k is defined by

$$\mathfrak{h}_k(x) = \prod_{j=1}^n \mathfrak{h}_{k_j}(x_j), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

The system $\{\mathfrak{h}_k\}_{k \in \mathbb{N}^n}$ is orthonormal and complete in $L^2(\mathbb{R}^n)$. Moreover, $\mathcal{H}\mathfrak{h}_k = (2|k| + n)\mathfrak{h}_k$, where $|k| = k_1 + \dots + k_n$ and $k = (k_1, \dots, k_n) \in \mathbb{N}^n$. The operator $-\mathcal{H}$ generates in $L^2(\mathbb{R}^n)$ the semigroup of operators $\{W_t^{\mathcal{H}}\}_{t>0}$ where, for every $t > 0$,

$$W_t^{\mathcal{H}}(f) = \sum_{k \in \mathbb{N}^n} e^{-t(2|k|+n)} c_k(f) \mathfrak{h}_k, \quad f \in L^2(\mathbb{R}^n),$$

being

$$c_k(f) = \int_{\mathbb{R}^n} \mathfrak{h}_k(y) f(y) dy, \quad k \in \mathbb{N}^n \text{ and } f \in L^2(\mathbb{R}^n).$$

According to the Mehler's formula ([33, (1.1.36)]) we can write, for every $t > 0$,

$$(7) \quad W_t^{\mathcal{H}}(f)(x) = \int_{\mathbb{R}^n} W_t^{\mathcal{H}}(x, y) f(y) dy, \quad f \in L^2(\mathbb{R}^n, \mathbb{B}),$$

where, for each $x, y \in \mathbb{R}^n$ and $t > 0$,

$$W_t^{\mathcal{H}}(x, y) = \frac{1}{\pi^{n/2}} \left(\frac{e^{-2t}}{1 - e^{-4t}} \right)^{n/2} \exp \left[-\frac{1}{4} \left(|x - y|^2 \frac{1 + e^{-2t}}{1 - e^{-2t}} + |x + y|^2 \frac{1 - e^{-2t}}{1 + e^{-2t}} \right) \right].$$

By defining $W_t^{\mathcal{H}}$, for every $t > 0$, on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, by means of (7), then the system $\{W_t^{\mathcal{H}}\}_{t>0}$ is a positive semigroup of contractions in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

Since $\{W_t^{\mathcal{L}}\}_{t>0}$ and $\{W_t^{\mathcal{H}}\}_{t>0}$ are positive, they have tensor extensions to $L^p(\mathbb{R}^n, \mathbb{B})$ satisfying the same L^p -boundedness properties.

If $\ell = 1, 2$ and $f \in L^p(\mathbb{R}^n, \mathbb{B})$, $1 < p < \infty$, we define

$$\mathcal{G}_{\mathcal{L}, \mathbb{B}}^{\ell}(f)(x, t) = t^{\ell} \partial_t^{\ell} W_t^{\mathcal{L}}(f)(x), \quad x \in \mathbb{R}^n, \quad t > 0, \quad n \geq 3,$$

and

$$\mathcal{G}_{\mathcal{H}, \mathbb{B}}^{\ell}(f)(x, t) = t^{\ell} \partial_t^{\ell} W_t^{\mathcal{H}}(f)(x), \quad x \in \mathbb{R}^n, \quad t > 0, \quad n \geq 1.$$

Let $\alpha > -1/2$. The Laguerre operator \mathcal{L}_{α} is defined by

$$\mathcal{L}_{\alpha} = \frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2 + \frac{\alpha^2 - 1/4}{x^2} \right), \quad x \in (0, \infty).$$

If $k \in \mathbb{N}$ we consider the k -th Laguerre function

$$\varphi_k^{\alpha}(x) = \left(\frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)} \right)^{1/2} e^{-x^2/2} x^{\alpha+1/2} L_k^{\alpha}(x^2), \quad x \in (0, \infty),$$

where L_k^α represents the k -th Laguerre polynomial ([31, pp. 100–102]). The family $\{\varphi_k^\alpha\}_{k \in \mathbb{N}}$ is orthonormal and complete in $L^2(0, \infty)$. Moreover, for every $k \in \mathbb{N}$,

$$\mathcal{L}_\alpha \varphi_k^\alpha = (2k + \alpha + 1) \varphi_k^\alpha.$$

The semigroup of operators $\{W_t^{\mathcal{L}_\alpha}\}_{t>0}$ generated by $-\mathcal{L}_\alpha$ in $L^2(0, \infty)$ is defined by

$$W_t^{\mathcal{L}_\alpha}(f) = \sum_{k=0}^{\infty} e^{-t(2k+\alpha+1)} c_k^\alpha(f) \varphi_k^\alpha, \quad t > 0 \text{ and } f \in L^2(0, \infty),$$

where $c_k^\alpha(f) = \int_0^\infty \varphi_k^\alpha(y) f(y) dy$, $k \in \mathbb{N}$.

According to the Mehler's formula ([33, (1.1.47)]) we can write, for every $t > 0$,

$$(8) \quad W_t^{\mathcal{L}_\alpha}(f)(x) = \int_0^\infty W_t^\alpha(x, y) f(y) dy, \quad f \in L^2(0, \infty),$$

where, for each $x, y, t \in (0, \infty)$

$$W_t^\alpha(x, y) = \left(\frac{2e^{-t}}{1 - e^{-2t}} \right)^{1/2} \left(\frac{2xye^{-t}}{1 - e^{-2t}} \right)^{1/2} I_\alpha \left(\frac{2xye^{-t}}{1 - e^{-2t}} \right) \exp \left[-\frac{1}{2}(x^2 + y^2) \frac{1 + e^{-2t}}{1 - e^{-2t}} \right],$$

and I_α denotes the modified Bessel function of the first kind and order α .

If we define, for every $t > 0$, $W_t^{\mathcal{L}_\alpha}$ on $L^p(0, \infty)$, $1 \leq p \leq \infty$ by (8), then $\{W_t^{\mathcal{L}_\alpha}\}_{t>0}$ is a positive semigroup of contractions in $L^p(0, \infty)$, $1 \leq p \leq \infty$. Moreover, for every $t > 0$, $W_t^{\mathcal{L}_\alpha}$ can be extended to $L^p((0, \infty), \mathbb{B})$ preserving the L^p -boundedness properties.

If $\ell = 1, 2$ we consider

$$\mathcal{G}_{\mathcal{L}_\alpha, \mathbb{B}}^\ell(f)(x, t) = t^\ell \partial_t^\ell W_t^{\mathcal{L}_\alpha}(f)(x), \quad x, t \in (0, \infty),$$

for every $f \in L^p((0, \infty), \mathbb{B})$, $1 < p < \infty$.

We now establish the main result of this paper.

Theorem 1.1. *Let \mathbb{B} be a Banach space and $\alpha > -1/2$. The following assertions are equivalent.*

(a) \mathbb{B} is UMD.

(b) For $\ell = 1, 2$ and for every (equivalently, for some) $1 < p < \infty$,

$$\|\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \sim \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}), \quad n \geq 1.$$

(c) For $\ell = 1, 2$ and for every (equivalently, for some) $1 < p < \infty$,

$$\|\mathcal{G}_{\mathcal{L}, \mathbb{B}}^\ell(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \sim \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}), \quad n \geq 3.$$

(d) For $\ell = 1, 2$ and for every (equivalently, for some) $1 < p < \infty$,

$$\|\mathcal{G}_{\mathcal{L}_\alpha, \mathbb{B}}^\ell(f)\|_{L^p((0, \infty), \gamma(H, \mathbb{B}))} \sim \|f\|_{L^p((0, \infty), \mathbb{B})}, \quad f \in L^p((0, \infty), \mathbb{B}).$$

Note that, since $\gamma(H, \mathbb{C}) = H$, the equivalences in Theorem 1.1, (b), (c) and (d) are Banach valued versions of the corresponding scalar equivalences (see [5], [32], [33, Chapter 4] and [38]).

In [1] we study square functions associated to the subordinated Poisson semigroup for the Hermite operator in a Banach valued setting. By using auxiliar operators and Cauchy-Riemann type equations adapted to the Hermite setting we characterized the UMD Banach spaces. We remark that, as it can be observed in [19], [25] and [39], in order to describe geometric properties of Banach spaces (UMD, q -martingale type and cotype,...) by using square functions, subordinated (Poisson) diffusion semigroups must be considered. Moreover, in [19], Hytönen dealt with diffusion semigroups and the semigroups $\{W_t^{\mathcal{H}}\}_{t>0}$, $\{W_t^{\mathcal{L}}\}_{t>0}$ and $\{W_t^{\mathcal{L}_\alpha}\}_{t>0}$ are not diffusion semigroups because they are not conservative. Then, in particular the results in [1] are not

covered by the ones in [19]. The results obtained by Hytönen for general diffusion semigroups in a UMD setting are weaker than the ones got for subordinated diffusion semigroups ([19, Theorem 5.1]). In order to get a better result for every diffusion semigroups Hytönen reduced the admissible class of Banach spaces. He considered the class of Banach spaces which are isomorphic to a closed subspace of a complex interpolation space $[Z, Y]_\theta$ where Z is a Hilbert space, Y is a UMD Banach space and $0 < \theta < 1$. We write ζ to refer this class of Banach spaces. ζ contains all the standard UMD spaces. In [27] Rubio de Francia posed the question whether the equality $\zeta = \text{UMD}$ holds. As far as we know this question remains open.

In contrast with the results in [19] we get Theorem 1.1 for the semigroups $\{W_t^{\mathcal{H}}\}_{t>0}$, $\{W_t^{\mathcal{L}}\}_{t>0}$ and $\{W_t^{\mathcal{L}_\alpha}\}_{t>0}$ which are not diffusion semigroups and, as it was above mentioned, they are not conservative. In order to prove Theorem 1.1 we use a procedure different to the one used in [19]. For establishing that if \mathbb{B} is a UMD Banach space the equivalences in (b), (c) and (d) hold, we take advantage of the following fact: close to singularities, our operators are good perturbations of the corresponding operators associated with the Laplacian operator. The exact meaning of this idea is clear in the proof. Then, we use [23, Theorem 4.2]. To see that the equivalences in (b), (c) and (d) imply that \mathbb{B} is UMD, we have taken into account that the UMD Banach spaces are characterized by the L^p -boundedness properties of the imaginary powers $\mathcal{H}^{i\gamma}$, $\mathcal{L}^{i\gamma}$, $\mathcal{L}_\alpha^{i\gamma}$, $\gamma > 0$ of \mathcal{H} , \mathcal{L} and \mathcal{L}_α , respectively ([2, Theorem 1.2] and [3, Theorem 3]).

In the next sections we prove our result for the Hermite operator in \mathbb{R}^n , $n \geq 1$ (Section 2), the Schrödinger operators in \mathbb{R}^n , $n \geq 3$ (Section 3) and the Laguerre operators in $(0, \infty)$ (Section 4).

Throughout this paper by C and c we always denote positive constants that can change in each occurrence.

Acknowledgements. The authors wish to thank Professor Peter Sjögren for posing us, after knowing our results in [1], the question of dealing with the heat semigroup for the Hermite operator.

2. PROOF OF THEOREM 1.1 FOR THE HERMITE OPERATOR

In this section we prove (a) \Leftrightarrow (b) in Theorem 1.1.

2.1. (a) \Rightarrow (b) Let $\ell = 1, 2$, $n \geq 1$ and $1 < p < \infty$. We define $\mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f)$, for every $f \in L^p(\mathbb{R}^n, \mathbb{B})$, as follows

$$\mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f)(x, t) = t^\ell \partial_t^\ell W_t(f)(x), \quad x \in \mathbb{R}^n \text{ and } t > 0.$$

Assume that \mathbb{B} is a UMD Banach space.

We start proving that

$$\|\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}).$$

Let $f \in L^p(\mathbb{R}^n, \mathbb{B})$. We can write

$$(9) \quad \partial_t^\ell W_t^{\mathcal{H}}(f)(x) = \int_{\mathbb{R}^n} \partial_t^\ell W_t^{\mathcal{H}}(x, y) f(y) dy, \quad x \in \mathbb{R}^n \text{ and } t > 0.$$

Derivation under the integral sign is justified. Indeed, we have, for every $x, y \in \mathbb{R}^n$ and $t > 0$,

$$(10) \quad \begin{aligned} \partial_t W_t^{\mathcal{H}}(x, y) &= \frac{1}{\pi^{n/2}} \left(\frac{e^{-2t}}{1 - e^{-4t}} \right)^{n/2} \exp \left[-\frac{1}{4} \left(|x - y|^2 \frac{1 + e^{-2t}}{1 - e^{-2t}} + |x + y|^2 \frac{1 - e^{-2t}}{1 + e^{-2t}} \right) \right] \\ &\quad \times \left[-n \frac{1 + e^{-4t}}{1 - e^{-4t}} + |x - y|^2 \frac{e^{-2t}}{(1 - e^{-2t})^2} - |x + y|^2 \frac{e^{-2t}}{(1 + e^{-2t})^2} \right], \end{aligned}$$

and

$$(11) \quad \begin{aligned} \partial_t^2 W_t^{\mathcal{H}}(x, y) &= \frac{1}{\pi^{n/2}} \left(\frac{e^{-2t}}{1 - e^{-4t}} \right)^{n/2} \exp \left[-\frac{1}{4} \left(|x - y|^2 \frac{1 + e^{-2t}}{1 - e^{-2t}} + |x + y|^2 \frac{1 - e^{-2t}}{1 + e^{-2t}} \right) \right] \\ &\quad \times \left\{ \left[-n \frac{1 + e^{-4t}}{1 - e^{-4t}} + |x - y|^2 \frac{e^{-2t}}{(1 - e^{-2t})^2} - |x + y|^2 \frac{e^{-2t}}{(1 + e^{-2t})^2} \right]^2 \right. \\ &\quad \left. + \frac{8ne^{-4t}}{(1 - e^{-4t})^2} - |x - y|^2 \frac{2e^{-2t}(1 + e^{-2t})}{(1 - e^{-2t})^3} + |x + y|^2 \frac{2e^{-2t}(1 - e^{-2t})}{(1 + e^{-2t})^3} \right\}. \end{aligned}$$

Hence, we deduce that, for $k = 0, 1, 2$,

$$(12) \quad |t^k \partial_t^k W_t^{\mathcal{H}}(x, y)| \leq C \frac{t^k e^{-nt} e^{-c|x-y|^2/t}}{(1 - e^{-2t})^{n/2+k}} \leq C \frac{e^{-c|x-y|^2/t}}{t^{n/2}}, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$

Estimation (12) justifies the derivation under the integral sign in (9).

We split the operators $\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell$ and $\mathcal{G}_{-\Delta, \mathbb{B}}^\ell$ as follows. We write, for $\mathcal{Q} = \mathcal{H}$ or $\mathcal{Q} = -\Delta$,

$$\mathcal{G}_{\mathcal{Q}, \mathbb{B}}^\ell = \mathcal{G}_{\mathcal{Q}, \mathbb{B}, \text{loc}}^\ell + \mathcal{G}_{\mathcal{Q}, \mathbb{B}, \text{glob}}^\ell,$$

where

$$(13) \quad \mathcal{G}_{\mathcal{Q}, \mathbb{B}, \text{loc}}^\ell(f)(x, t) = \mathcal{G}_{\mathcal{Q}, \mathbb{B}}^\ell(\chi_{B(x, \rho(x))}(y) f(y))(x, t), \quad x \in \mathbb{R}^n, t > 0,$$

and

$$\rho(x) = \begin{cases} \frac{1}{2}, & |x| \leq 1 \\ \frac{1}{1 + |x|}, & |x| > 1 \end{cases}.$$

For every $x \in \mathbb{R}^n$, $\rho(x)$ is called the critical radius in x (see [28, p. 516]).

We consider the following decomposition of the operator $\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell$:

$$\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell = \sum_{j=1}^3 T_{j, \mathbb{B}}^\ell,$$

where $T_{1, \mathbb{B}}^\ell = \mathcal{G}_{\mathcal{H}, \mathbb{B}, \text{loc}}^\ell - \mathcal{G}_{-\Delta, \mathbb{B}, \text{loc}}^\ell$, $T_{2, \mathbb{B}}^\ell = \mathcal{G}_{\mathcal{H}, \mathbb{B}, \text{glob}}^\ell$ and $T_{3, \mathbb{B}}^\ell = \mathcal{G}_{-\Delta, \mathbb{B}, \text{loc}}^\ell$.

Lemma 2.1. *Let \mathbb{B} be a UMD Banach space and $j = 1, 2, 3$. Then, there exists $C > 0$ verifying that*

$$(14) \quad \|T_{j, \mathbb{B}}^\ell(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}).$$

Proof of Lemma 2.1 for $T_{3, \mathbb{B}}^\ell$. We consider $\varphi^\ell(x) = \left(\partial_t^\ell G_{\sqrt{t}}(x) \right)_{|t=1}$, $x \in \mathbb{R}^n$. Thus, $\varphi^\ell \in S(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, where $S(\mathbb{R}^n)$ denotes the Schwartz class. Moreover, φ^ℓ satisfies conditions

(C1) and (C2) in [23, p. 111]. Indeed, according to [13, p. 121, (23)] we have that

$$\begin{aligned}\widehat{\varphi}^\ell(y) &= \int_{\mathbb{R}^n} e^{-ix \cdot y} \left[\partial_t^\ell \left(\frac{e^{-|x|^2/(4t)}}{(4\pi t)^{n/2}} \right) \right] \Big|_{t=1} dx \\ &= \partial_t^\ell \left[\int_{\mathbb{R}^n} e^{-ix \cdot y} \frac{e^{-|x|^2/(4t)}}{(4\pi t)^{n/2}} dx \right] \Big|_{t=1} \\ &= \partial_t^\ell \left(e^{-t|y|^2} \right) \Big|_{t=1} = (-|y|^2)^\ell e^{-|y|^2}, \quad y \in \mathbb{R}^n.\end{aligned}$$

Now, straightforward manipulations allow us to see that the conditions (C1) and (C2) in [23, p. 111] are satisfied by φ^ℓ . On the other hand, $\varphi_t^\ell(x) = t^{-n} \varphi^\ell(x/t) = \left(s^\ell \partial_s^\ell G_{\sqrt{s}}(x) \right) \Big|_{s=t^2}$, $x \in \mathbb{R}^n$, and $t > 0$. Note that if $\{h_n\}_{n \in \mathbb{N}}$ is an orthonormal basis in H , then $\{h_n(\sqrt{t})/\sqrt{2}\}_{n \in \mathbb{N}}$ is also an orthonormal basis in H . Hence,

$$\|\mathcal{G}_{-\Delta, \mathbb{B}}^\ell(g)(x, \cdot)\|_{\gamma(H, \mathbb{B})} = \sqrt{2} \|(\varphi^\ell * g)(x)\|_{\gamma(H, \mathbb{B})}, \quad g \in S(\mathbb{R}^n, \mathbb{B}) \text{ and } x \in \mathbb{R}^n.$$

Hence, by invoking [23, Theorem 4.2] there exists a bounded operator $\tilde{\mathcal{G}}_{-\Delta, \mathbb{B}}^\ell$ from $L^p(\mathbb{R}^n, \mathbb{B})$ into $L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))$ such that

$$\tilde{\mathcal{G}}_{-\Delta, \mathbb{B}}^\ell(g) = \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(g), \quad g \in S(\mathbb{R}^n, \mathbb{B}).$$

Let $f \in L^p(\mathbb{R}^n, \mathbb{B})$. We are going to see that $\tilde{\mathcal{G}}_{-\Delta, \mathbb{B}}^\ell(f) = \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f)$. In order to do this we choose a sequence $(f_m)_{m=1}^\infty \subset C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$ such that $f_m \rightarrow f$, as $m \rightarrow \infty$, in $L^p(\mathbb{R}^n, \mathbb{B})$. Note that $C_c^\infty(\mathbb{R}^n) \otimes \mathbb{B} \subset S(\mathbb{R}^n, \mathbb{B})$ is a dense subset of $L^p(\mathbb{R}^n, \mathbb{B})$. It can be shown that

$$(15) \quad |t^\ell \partial_t^\ell G_{\sqrt{t}}(x - y)| \leq C \frac{e^{-c|x-y|^2/t}}{t^{n/2}}, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$

Then, for every $N \in \mathbb{N}$ and $x \in \mathbb{R}^n$, there exists $C_N > 0$ for which

$$\begin{aligned}\|\mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f)(x, \cdot) - \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f_m)(x, \cdot)\|_{L^2((1/N, N), \frac{dt}{t}; \mathbb{B})} &\leq \int_{\mathbb{R}^n} \|f(y) - f_m(y)\|_{\mathbb{B}} \|t^\ell \partial_t^\ell G_{\sqrt{t}}(x - y)\|_{L^2((1/N, N), \frac{dt}{t})} dy \\ &\leq C_N \int_{\mathbb{R}^n} \|f(y) - f_m(y)\|_{\mathbb{B}} \left(\int_{1/N}^N \frac{1}{(t + |x - y|^2)^{n+1}} dt \right)^{1/2} dy \\ &\leq C_N \int_{\mathbb{R}^n} \|f(y) - f_m(y)\|_{\mathbb{B}} \frac{1}{(1/N + |x - y|^2)^{n/2}} dy \\ &\leq C_N \|f - f_m\|_{L^p(\mathbb{R}^n, \mathbb{B})} \left(\int_{\mathbb{R}^n} \frac{1}{(1/\sqrt{N} + |x - y|)^{np'}} dy \right)^{1/p'} \\ &\leq C_N \|f - f_m\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad m \in \mathbb{N}.\end{aligned}$$

Hence, for every $N \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$\mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f_m)(x, \cdot) \rightarrow \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f)(x, \cdot), \quad \text{as } m \rightarrow \infty \text{ in } L^2\left((1/N, N), \frac{dt}{t}; \mathbb{B}\right).$$

On the other hand,

$$\mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f_m) \rightarrow \tilde{\mathcal{G}}_{-\Delta, \mathbb{B}}^\ell(f), \quad \text{as } m \rightarrow \infty \text{ in } L^p(\mathbb{R}^n, \gamma(H, \mathbb{B})).$$

Then, there exists a subsequence of $(f_m)_{m=1}^\infty$ which we continue denoting by $(f_m)_{m=1}^\infty$, satisfying

$$\mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f_m)(x, \cdot) \rightarrow \tilde{\mathcal{G}}_{-\Delta, \mathbb{B}}^\ell(f)(x), \quad \text{as } m \rightarrow \infty \text{ in } \gamma(H, \mathbb{B}),$$

for every $x \in \mathbf{N}$, where $\mathbf{N} \subset \mathbb{R}^n$ and $|\mathbb{R}^n \setminus \mathbf{N}| = 0$. Since $\gamma(H, \mathbb{B})$ is continuously contained in $L(H, \mathbb{B})$, we have that, for every $x \in \mathbf{N}$,

$$\mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f_m)(x, \cdot) \longrightarrow \tilde{\mathcal{G}}_{-\Delta, \mathbb{B}}^\ell(f)(x), \quad \text{as } m \rightarrow \infty \text{ in } L(H, \mathbb{B}).$$

Let $x \in \mathbf{N}$. We choose $h \in H$ such that its support is compact and contained in $(0, \infty)$. For every $S \in \mathbb{B}^*$ we can write

$$\begin{aligned} \langle S, [\tilde{\mathcal{G}}_{-\Delta, \mathbb{B}}^\ell(f)(x)](h) \rangle_{\mathbb{B}^*, \mathbb{B}} &= \lim_{m \rightarrow \infty} \langle S, [\mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f_m)(x, \cdot)](h) \rangle_{\mathbb{B}^*, \mathbb{B}} \\ &= \langle S, \int_0^\infty \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f)(x, t) h(t) \frac{dt}{t} \rangle_{\mathbb{B}^*, \mathbb{B}} = \int_0^\infty \langle S, \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f)(x, t) \rangle_{\mathbb{B}^*, \mathbb{B}} h(t) \frac{dt}{t}. \end{aligned}$$

Moreover,

$$\begin{aligned} \left| \int_0^\infty \langle S, \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f)(x, t) \rangle_{\mathbb{B}^*, \mathbb{B}} h(t) \frac{dt}{t} \right| &= \left| \langle S, [\tilde{\mathcal{G}}_{-\Delta, \mathbb{B}}^\ell(f)(x)](h) \rangle_{\mathbb{B}^*, \mathbb{B}} \right| \\ &\leq \|S\|_{\mathbb{B}^*} \|\tilde{\mathcal{G}}_{-\Delta, \mathbb{B}}^\ell(f)(x)\|_{L(H, \mathbb{B})} \|h\|_{\mathbb{B}}. \end{aligned}$$

We conclude that $\langle S, \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f)(x, \cdot) \rangle_{\mathbb{B}^*, \mathbb{B}} \in H$ and

$$\langle S, [\tilde{\mathcal{G}}_{-\Delta, \mathbb{B}}^\ell(f)(x)](w) \rangle_{\mathbb{B}^*, \mathbb{B}} = \int_0^\infty \langle S, \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f)(x, t) \rangle_{\mathbb{B}^*, \mathbb{B}} w(t) \frac{dt}{t}, \quad w \in H.$$

Thus we prove that $\tilde{\mathcal{G}}_{-\Delta, \mathbb{B}}^\ell(f)(x) = \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(f)(x, \cdot)$ as elements of $\gamma(H, \mathbb{B})$.

We now use the ideas developed in [17, Proposition 2.3] to see that (14) holds for $j = 2$. According to [9, Proposition 5], for every $M > 0$ there exists $C > 0$ such that

$$(16) \quad \frac{1}{C} \leq \frac{\rho(x)}{\rho(y)} \leq C, \quad x \in B(y, M\rho(y)).$$

We can find a sequence $(x_k)_{k=1}^\infty$ such that

- (i) $\bigcup_{k=1}^\infty B(x_k, \rho(x_k)) = \mathbb{R}^n$,
- (ii) For every $M > 0$ there exists $m \in \mathbb{N}$ such that, for each $j \in \mathbb{N}$,

$$\text{card} \{k \in \mathbb{N} : B(x_k, M\rho(x_k)) \cap B(x_j, M\rho(x_j)) \neq \emptyset\} \leq m.$$

Let $k \in \mathbb{N}$. If $x \in B(x_k, \rho(x_k))$, then (16) implies that $|y - x_k| \leq \rho(x) + \rho(x_k) \leq C_0\rho(x_k)$, provided that $y \in B(x, \rho(x))$. Here $C_0 > 0$ does not depend on $k \in \mathbb{N}$. We can write for every $x \in B(x_k, \rho(x_k))$ and $t > 0$,

$$\begin{aligned} \mathcal{G}_{-\Delta, \mathbb{B}, \text{loc}}^\ell(f)(x, t) &= \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(\chi_{B(x_k, C_0\rho(x_k))} f)(x, t) + \mathcal{G}_{-\Delta, \mathbb{B}}^\ell((\chi_{B(x, \rho(x))} - \chi_{B(x_k, C_0\rho(x_k))}) f)(x, t) \\ &= \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(\chi_{B(x_k, C_0\rho(x_k))} f)(x, t) - \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(\chi_{B(x_k, C_0\rho(x_k)) \setminus B(x, \rho(x))} f)(x, t). \end{aligned}$$

Let $x \in B(x_k, \rho(x_k))$. We consider the operator

$$L_{k,x}(f)(t) = \mathcal{G}_{-\Delta, \mathbb{B}}^\ell(\chi_{B(x_k, C_0\rho(x_k)) \setminus B(x, \rho(x))} f)(x, t), \quad t > 0.$$

By (15) we have that

$$\|t^\ell \partial_t^\ell G_{\sqrt{t}}(x - y)\|_H \leq C \left(\int_0^\infty \frac{e^{-c|x-y|^2/t}}{t^{n+1}} dt \right)^{1/2} \leq \frac{C}{|x-y|^n}, \quad y \in \mathbb{R}^n \setminus \{x\}.$$

Hence, for every $y \notin B(x, \rho(x))$, the function $g_{x,y}(t) = t^\ell \partial_t^\ell G_{\sqrt{t}}(x-y)$, $t \in (0, \infty)$, belongs to H and $\|g_{x,y}\|_H \leq C/\rho(x)^n$. Then, $L_{k,x}(f) \in L^2((0, \infty), dt/t; \mathbb{B})$ and

$$\begin{aligned} \|L_{k,x}(f)\|_{L^2((0, \infty), \frac{dt}{t}; \mathbb{B})} &\leq \int_{B(x_k, C_0 \rho(x_k)) \setminus B(x, \rho(x))} \|g_{x,y}\|_H \|f(y)\|_{\mathbb{B}} dy \\ &\leq \frac{C}{\rho(x)^n} \int_{B(x_k, C_0 \rho(x_k))} \|f(y)\|_{\mathbb{B}} dy. \end{aligned}$$

Hence, $L_{k,x}(f) \in \gamma(H, \mathbb{B})$. Indeed, by (3) we have

$$\|L_{k,x}(f)\|_{\gamma(H, \mathbb{B})} = \sup \left(\mathbb{E} \left\| \sum_j \gamma_j \int_0^\infty L_{k,x}(f)(t) h_j(t) \frac{dt}{t} \right\|_{\mathbb{B}}^2 \right)^{1/2},$$

where $(\gamma_j)_{j=1}^\infty$ is a sequence of independent standard Gaussian random variables and the supremum is taken over all the finite families $\{h_j\}$ of orthonormal functions in H . Suppose that $(h_j)_{j=1}^m$ is an orthonormal set in H . We can write

$$\begin{aligned} &\left(\mathbb{E} \left\| \sum_{j=1}^m \gamma_j \int_0^\infty L_{k,x}(f)(t) h_j(t) \frac{dt}{t} \right\|_{\mathbb{B}}^2 \right)^{1/2} \\ &= \left(\mathbb{E} \left\| \sum_{j=1}^m \gamma_j \int_0^\infty \int_{B(x_k, C_0 \rho(x_k)) \setminus B(x, \rho(x))} g_{x,y}(t) f(y) dy h_j(t) \frac{dt}{t} \right\|_{\mathbb{B}}^2 \right)^{1/2} \\ &= \left(\mathbb{E} \left\| \int_{B(x_k, C_0 \rho(x_k)) \setminus B(x, \rho(x))} f(y) \sum_{j=1}^m \gamma_j \int_0^\infty g_{x,y}(t) h_j(t) \frac{dt}{t} dy \right\|_{\mathbb{B}}^2 \right)^{1/2} \\ &\leq \int_{B(x_k, C_0 \rho(x_k)) \setminus B(x, \rho(x))} \|f(y)\|_{\mathbb{B}} \left(\mathbb{E} \left\| \sum_{j=1}^m \gamma_j \int_0^\infty g_{x,y}(t) h_j(t) \frac{dt}{t} \right\|_{\mathbb{B}}^2 \right)^{1/2} dy \\ &\leq \int_{B(x_k, C_0 \rho(x_k)) \setminus B(x, \rho(x))} \|f(y)\|_{\mathbb{B}} \|g_{x,y}\|_{\gamma(H, \mathbb{C})} dy \\ &= \int_{B(x_k, C_0 \rho(x_k)) \setminus B(x, \rho(x))} \|f(y)\|_{\mathbb{B}} \|g_{x,y}\|_H dy \\ &\leq \frac{C}{\rho(x)^n} \int_{B(x_k, C_0 \rho(x_k))} \|f(y)\|_{\mathbb{B}} dy. \end{aligned}$$

Hence,

$$\|L_{k,x}(f)\|_{\gamma(H, \mathbb{B})} \leq \frac{C}{\rho(x)^n} \int_{B(x_k, C_0 \rho(x_k))} \|f(y)\|_{\mathbb{B}} dy.$$

By using (16), we deduce that $B(x_k, C_0 \rho(x_k)) \subset B(x, C_1 \rho(x))$, where C_1 does not depend on k neither on x . Then, we get

$$\|L_{k,x}(f)\|_{\gamma(H, \mathbb{B})} \leq \frac{C}{\rho(x)^n} \int_{B(x, C_1 \rho(x))} \|f(y)\|_{\mathbb{B}} dy \leq \mathcal{M}(\|f\|_{\mathbb{B}})(x),$$

where \mathcal{M} denotes the Hardy-Littlewood maximal function.

According to the classical maximal theorem and the boundedness of $\mathcal{G}_{-\Delta, \mathbb{B}}^\ell$ from $L^p(\mathbb{R}^n, \mathbb{B})$ into $L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, we obtain

$$\begin{aligned}
 \|\mathcal{G}_{-\Delta, \mathbb{B}, \text{loc}}^\ell(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))}^p &\leq \sum_{k=1}^{\infty} \int_{B(x_k, \rho(x_k))} \|\mathcal{G}_{-\Delta, \mathbb{B}, \text{loc}}^\ell(f)\|_{\gamma(H, \mathbb{B})}^p dx \\
 &\leq C \sum_{k=1}^{\infty} \left(\int_{\mathbb{R}^n} \|\mathcal{G}_{-\Delta, \mathbb{B}}^\ell(\chi_{B(x_k, C_0 \rho(x_k))} f)(x, \cdot)\|_{\gamma(H, \mathbb{B})}^p dx \right. \\
 &\quad \left. + \int_{\mathbb{R}^n} \|\mathcal{G}_{-\Delta, \mathbb{B}}^\ell(\chi_{B(x_k, C_0 \rho(x_k)) \setminus B(x, \rho(x))} f)(x, \cdot)\|_{\gamma(H, \mathbb{B})}^p dx \right) \\
 &\leq C \left(\sum_{k=1}^{\infty} \int_{B(x_k, C_0 \rho(x_k))} \|f(y)\|_{\mathbb{B}}^p dy + \int_{\mathbb{R}^n} |\mathcal{M}(\|f\|_{\mathbb{B}})(x)|^p dx \right) \\
 &\leq C \int_{\mathbb{R}^n} \|f(y)\|_{\mathbb{B}}^p dy.
 \end{aligned}$$

We conclude that (14) holds for $T_{3, \mathbb{B}}^\ell$. □

Proof of Lemma 2.1 for $T_{1, \mathbb{B}}^\ell$. By using the perturbation formula ([9, (5.25)]) we can write

$$\begin{aligned}
 \partial_t [G_{\sqrt{t}}(x-y) - W_t^{\mathcal{H}}(x, y)] &= \int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 [\partial_u G_{\sqrt{u}}(x-z)]|_{u=t-s} W_s^{\mathcal{H}}(z, y) dz ds \\
 &\quad + \int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 G_{\sqrt{s}}(x-z) [\partial_u W_u^{\mathcal{H}}(z, y)]|_{u=t-s} dz ds \\
 &\quad + \int_{\mathbb{R}^n} |z|^2 G_{\sqrt{t/2}}(x-z) W_{t/2}^{\mathcal{H}}(z, y) dz \\
 &= H_1^1(x, y, t) + H_2^1(x, y, t) + H_3^1(x, y, t), \quad x, y \in \mathbb{R}^n \text{ and } t > 0.
 \end{aligned}$$

Then, it follows that

$$\begin{aligned}
 \partial_t^2 [G_{\sqrt{t}}(x-y) - W_t^{\mathcal{H}}(x, y)] &= \int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 [\partial_u^2 G_{\sqrt{u}}(x-z)]|_{u=t-s} W_s^{\mathcal{H}}(z, y) dz ds \\
 &\quad + \int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 G_{\sqrt{s}}(x-z) [\partial_u^2 W_u^{\mathcal{H}}(z, y)]|_{u=t-s} dz ds \\
 &\quad + \int_{\mathbb{R}^n} |z|^2 \left([\partial_u G_{\sqrt{u}}(x-z)]|_{u=t/2} W_{t/2}^{\mathcal{H}}(z, y) + G_{\sqrt{t/2}}(x-z) [\partial_u W_u^{\mathcal{H}}(z, y)]|_{u=t/2} \right) dz \\
 &= H_1^2(x, y, t) + H_2^2(x, y, t) + H_3^2(x, y, t), \quad x, y \in \mathbb{R}^n \text{ and } t > 0.
 \end{aligned}$$

The following estimation will be very useful in the sequel. For every $x \in \mathbb{R}^n$ and $t > 0$, we get

$$\int_{\mathbb{R}^n} e^{-c|x-y|^2/t} |y|^2 dy \leq C \int_{\mathbb{R}^n} e^{-c|z|^2/t} (|z|^2 + |x|^2) dz \leq C t^{n/2} (t + |x|^2).$$

Then, we obtain

$$(17) \quad \int_{\mathbb{R}^n} e^{-c|x-y|^2/t} |y|^2 dy \leq C \frac{t^{n/2}}{\rho(x)^2}, \quad x \in \mathbb{R}^n \text{ and } 0 < t \leq \rho(x)^2.$$

Minkowski's inequality leads to

$$\begin{aligned}
 \|T_{1, \mathbb{B}}^\ell(f)(x, \cdot)\|_{L^2((0, \infty), \frac{dt}{t}; \mathbb{B})} &\leq \int_{B(x, \rho(x))} \|f(y)\|_{\mathbb{B}} \left(\int_0^\infty |t^\ell \partial_t^\ell [G_{\sqrt{t}}(x-y) - W_t^{\mathcal{H}}(x, y)]|^2 \frac{dt}{t} \right)^{1/2} dy \\
 &\leq C \sum_{j=1}^3 \int_{B(x, \rho(x))} \|f(y)\|_{\mathbb{B}} \left(\int_0^\infty |t^\ell H_j^\ell(x, y, t)|^2 \frac{dt}{t} \right)^{1/2} dy, \quad x \in \mathbb{R}^n.
 \end{aligned}$$

We now study

$$A_j^\ell(x, y) = \left(\int_0^{\rho(x)^2} |t^\ell H_j^\ell(x, y, t)|^2 \frac{dt}{t} \right)^{1/2}, \quad x, y \in \mathbb{R}^n \text{ and } j = 1, 2, 3.$$

According to (12), (15), (16) and (17) we get

$$\begin{aligned} \left(\int_0^{\rho(x)^2} |t^\ell H_1^\ell(x, y, t)|^2 \frac{dt}{t} \right)^{1/2} &\leq C \left(\int_0^{\rho(x)^2} \left(\int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 \frac{e^{-c|x-z|^2/(t-s)}}{(t-s)^{n/2}} \frac{e^{-c|y-z|^2/s}}{s^{n/2}} dz ds \right)^2 \frac{dt}{t} \right)^{1/2} \\ &\leq C \left(\int_0^{\rho(x)^2} \frac{e^{-c(|x-y|^2+|y-z|^2)/t}}{t^{n+1}} \left(\int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 \frac{e^{-c|y-z|^2/s}}{s^{n/2}} dz ds \right)^2 dt \right)^{1/2} \\ &\leq \frac{C}{\rho(x)^2} \left(\int_0^{\rho(x)^2} \frac{e^{-c|x-y|^2/t}}{t^{n-1}} dt \right)^{1/2} \leq \frac{C}{\rho(x)^2 |x-y|^{n-1/2}} \left(\int_0^{\rho(x)^2} \sqrt{t} dt \right)^{1/2} \\ &\leq \frac{C}{\sqrt{\rho(x)} |x-y|^{n-1/2}}, \quad x \in \mathbb{R}^n, y \in B(x, \rho(x)), x \neq y. \end{aligned}$$

Also by taking into account (12) and again (17) it follows that

$$\begin{aligned} \left(\int_0^{\rho(x)^2} |t^\ell H_2^\ell(x, y, t)|^2 \frac{dt}{t} \right)^{1/2} &\leq C \left(\int_0^{\rho(x)^2} \left(\int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 \frac{e^{-c|x-z|^2/s}}{s^{n/2}} \frac{e^{-c|y-z|^2/(t-s)}}{(t-s)^{n/2}} dz ds \right)^2 \frac{dt}{t} \right)^{1/2} \\ &\leq C \left(\int_0^{\rho(x)^2} \frac{e^{-c(|x-y|^2+|y-z|^2)/t}}{t^{n+1}} \left(\int_0^{t/2} \int_{\mathbb{R}^n} |z|^2 \frac{e^{-c|x-z|^2/s}}{s^{n/2}} dz ds \right)^2 dt \right)^{1/2} \\ &\leq \frac{C}{\sqrt{\rho(x)} |x-y|^{n-1/2}}, \quad x, y \in \mathbb{R}^n, x \neq y, \end{aligned}$$

and

$$\begin{aligned} \left(\int_0^{\rho(x)^2} |t^\ell H_3^\ell(x, y, t)|^2 \frac{dt}{t} \right)^{1/2} &\leq C \left(\int_0^{\rho(x)^2} \left(\int_{\mathbb{R}^n} |z|^2 \frac{e^{-c(|x-z|^2+|y-z|^2)/t}}{t^{n-1}} dz \right)^2 \frac{dt}{t} \right)^{1/2} \\ &\leq C \left(\int_0^{\rho(x)^2} \frac{e^{-c|x-y|^2/t}}{t^{2n-1}} \left(\int_{\mathbb{R}^n} |z|^2 e^{-c|x-z|^2/t} dz \right)^2 dt \right)^{1/2} \\ &\leq \frac{C}{\rho(x)^2} \left(\int_0^{\rho(x)^2} \frac{e^{-c|x-y|^2/t}}{t^{n-1}} dt \right)^{1/2} \leq \frac{C}{\sqrt{\rho(x)} |x-y|^{n-1/2}}, \quad x, y \in \mathbb{R}^n, x \neq y. \end{aligned}$$

By combining the above estimations we obtain

$$\begin{aligned} \sum_{j=1}^3 \int_{B(x, \rho(x))} \|f(y)\|_{\mathbb{B}} A_j^\ell(x, y) dy &\leq C \int_{B(x, \rho(x))} \frac{\|f(y)\|_{\mathbb{B}}}{\sqrt{\rho(x)} |x-y|^{n-1/2}} dy \\ &\leq C \sum_{m=0}^{\infty} \frac{1}{\sqrt{\rho(x)}} \int_{2^{-m-1}\rho(x) \leq |x-y| < 2^{-m}\rho(x)} \frac{\|f(y)\|_{\mathbb{B}}}{|x-y|^{n-1/2}} dy \\ &\leq C \sum_{m=0}^{\infty} \frac{1}{\rho(x)^n 2^{-m(n-1/2)}} \int_{B(x, 2^{-m}\rho(x))} \|f(y)\|_{\mathbb{B}} dy \\ &\leq C \sum_{m=0}^{\infty} \frac{1}{2^{m/2}} \mathcal{M}(\|f\|_{\mathbb{B}})(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{18}$$

On the other hand, (12) and (15) lead to

$$\begin{aligned}
 & \int_{B(x, \rho(x))} \|f(y)\|_{\mathbb{B}} \left(\int_{\rho(x)^2}^{\infty} |t^\ell \partial_t^\ell [G_{\sqrt{t}}(x-y) - W_t^{\mathcal{H}}(x, y)]|^2 \frac{dt}{t} \right)^{1/2} dy \\
 & \leq C \int_{B(x, \rho(x))} \|f(y)\|_{\mathbb{B}} \left(\int_{\rho(x)^2}^{\infty} \frac{e^{-c|x-y|^2/t}}{t^{n+1}} dt \right)^{1/2} dy \\
 & \leq C \left(\int_{\rho(x)^2}^{\infty} \frac{1}{t^{n+1}} dt \right)^{1/2} \int_{B(x, \rho(x))} \|f(y)\|_{\mathbb{B}} dy \leq \frac{1}{\rho(x)^n} \int_{B(x, \rho(x))} \|f(y)\|_{\mathbb{B}} dy \\
 (19) \quad & \leq C \mathcal{M}(\|f\|_{\mathbb{B}})(x), \quad x \in \mathbb{R}^n.
 \end{aligned}$$

From (18), (19) we conclude that

$$\|T_{1, \mathbb{B}}^\ell(f)(x, \cdot)\|_{L^2((0, \infty), \frac{dt}{t}; \mathbb{B})} \leq C \mathcal{M}(\|f\|_{\mathbb{B}})(x), \quad x \in \mathbb{R}^n.$$

Then, $T_{1, \mathbb{B}}^\ell(f)(x, \cdot) \in \gamma(H, \mathbb{B})$ and by proceeding as above we show that

$$\|T_{1, \mathbb{B}}^\ell(f)(x, \cdot)\|_{\gamma(H, \mathbb{B})} \leq C \mathcal{M}(\|f\|_{\mathbb{B}})(x), \quad x \in \mathbb{R}^n.$$

Classical maximal theorems leads to

$$\|T_{1, \mathbb{B}}^\ell(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}.$$

□

Proof of Lemma 2.1 for $T_{2, \mathbb{B}}^\ell$. By taking into account the following estimation (see [4, (4.4) and (4.5)])

$$\exp \left[-c \left(|x-y|^2 \frac{1+e^{-2t}}{1-e^{-2t}} + |x+y|^2 \frac{1-e^{-2t}}{1+e^{-2t}} \right) \right] \leq C \exp [-c(|x|+|y|)|x-y|], \quad x, y \in \mathbb{R}^n \text{ and } t > 0,$$

we get, by using (10) and (11),

$$|t^\ell \partial_t^\ell W_t^{\mathcal{H}}(x, y)| \leq C \frac{e^{-c(|x|+|y|)|x-y|} e^{-c|x-y|^2/t}}{t^{n/2}}, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$

Hence, Minkowski's inequality allows us to write

$$\begin{aligned}
 \|T_{2, \mathbb{B}}^\ell(f)(x, \cdot)\|_{L^2((0, \infty), \frac{dt}{t}; \mathbb{B})} & \leq \int_{|x-y| > \rho(x)} \|f(y)\|_{\mathbb{B}} \left(\int_0^\infty |t^\ell \partial_t^\ell W_t^{\mathcal{H}}(x, y)|^2 \frac{dt}{t} \right)^{1/2} dy \\
 & \leq C \int_{|x-y| > \rho(x)} \|f(y)\|_{\mathbb{B}} e^{-c(|x|+|y|)|x-y|} \left(\int_0^\infty \frac{e^{-c|x-y|^2/t}}{t^{n+1}} dt \right)^{1/2} dy \\
 & \leq C \int_{|x-y| > \rho(x)} \|f(y)\|_{\mathbb{B}} \frac{e^{-c(|x|+|y|)|x-y|}}{|x-y|^n} dy \\
 & \leq C \sum_{m=0}^{\infty} \frac{1}{(2^m \rho(x))^n} \int_{2^m \rho(x) < |x-y| \leq 2^{m+1} \rho(x)} \|f(y)\|_{\mathbb{B}} e^{-c(|x|+|y|)2^m \rho(x)} dy, \quad x \in \mathbb{R}^n.
 \end{aligned}$$

Note that if $|x-y| > \rho(x)$, then

$$(|x|+|y|)\rho(x) \geq |x-y|\rho(x) > \rho(x)^2 = \frac{1}{4}, \quad \text{when } |x| \leq 1,$$

and

$$(|x|+|y|)\rho(x) \geq \frac{|x|}{1+|x|} > \frac{1}{2}, \quad \text{when } |x| > 1.$$

Hence,

$$\begin{aligned} \|T_{2,\mathbb{B}}^\ell(f)(x, \cdot)\|_{L^2((0,\infty), \frac{dt}{t}; \mathbb{B})} &\leq C \sum_{m=0}^{\infty} \frac{e^{-c2^m}}{(2^m \rho(x))^n} \int_{|x-y| \leq 2^{m+1} \rho(x)} \|f(y)\|_{\mathbb{B}} dy \\ &\leq C \mathcal{M}(\|f\|_{\mathbb{B}})(x), \quad x \in \mathbb{R}^n, \end{aligned}$$

and we get that $T_{2,\mathbb{B}}^\ell(f)(x, \cdot) \in \gamma(H, \mathbb{B})$, $x \in \mathbb{R}^n$, and

$$\|T_{2,\mathbb{B}}^\ell(f)(x, \cdot)\|_{\gamma(H, \mathbb{B})} \leq C \mathcal{M}(\|f\|_{\mathbb{B}})(x), \quad x \in \mathbb{R}^n.$$

Maximal theorem implies now that (14) holds for $T_{2,\mathbb{B}}^\ell$. \square

We conclude that there exists $C > 0$ independent of f for which

$$(20) \quad \|\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}.$$

Our next objective is to establish that

$$\|f\|_{L^p(\mathbb{R}^n, \mathbb{B})} \leq C \|\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))},$$

where $C > 0$ does not depend on f .

In order to show this, we prove the following polarization formula.

Proposition 2.1. *Let \mathbb{B} be a UMD Banach space, $1 < q < \infty$ and $k \in \mathbb{N}$. For every $f \in L^q(\mathbb{R}^n) \otimes \mathbb{B}$ and $g \in L^{q'}(\mathbb{R}^n) \otimes \mathbb{B}^*$, we have that*

$$(21) \quad \int_{\mathbb{R}^n} \int_0^\infty \langle t^k \partial_t^k W_t^{\mathcal{H}}(g)(x), t^k \partial_t^k W_t^{\mathcal{H}}(f)(x) \rangle_{\mathbb{B}^*, \mathbb{B}} \frac{dt dx}{t} = \frac{\Gamma(2k)}{2^{2k}} \int_{\mathbb{R}^n} \langle g(x), f(x) \rangle_{\mathbb{B}^*, \mathbb{B}} dx.$$

Proof. This property can be proved by using standard spectral arguments. Indeed, if $f, g \in \text{span}\{\mathfrak{h}_m\}_{m \in \mathbb{N}^n}$, we have that

$$\int_{\mathbb{R}^n} \int_0^\infty t^k \partial_t^k W_t^{\mathcal{H}}(g)(x) t^k \partial_t^k W_t^{\mathcal{H}}(f)(x) \frac{dt}{t} dx = \frac{\Gamma(2k)}{2^{2k}} \int_{\mathbb{R}^n} g(x) f(x) dx.$$

Since $\text{span}\{\mathfrak{h}_m\}_{m \in \mathbb{N}^n}$ is dense in $L^q(\mathbb{R}^n)$, by taking into account that, for every $1 < r < \infty$,

$$\|g^k(\{W_t^{\mathcal{H}}\}_{t>0})(f)\|_{L^r(\mathbb{R}^n)} = \|\mathcal{G}_{\mathcal{H}, \mathbb{C}}^k(f)\|_{L^r(\mathbb{R}^n, H)} \leq C \|f\|_{L^r(\mathbb{R}^n)}, \quad f \in L^r(\mathbb{R}^n),$$

we conclude that

$$(22) \quad \int_{\mathbb{R}^n} \int_0^\infty t^k \partial_t^k W_t^{\mathcal{H}}(f)(x) t^k \partial_t^k W_t^{\mathcal{H}}(g)(x) \frac{dt}{t} = \frac{\Gamma(2k)}{2^{2k}} \int_{\mathbb{R}^n} f(x) g(x) dx,$$

for every $f \in L^q(\mathbb{R}^n)$ and $g \in L^{q'}(\mathbb{R}^n)$.

From (22) we can immediately deduce that (21) holds for every $L^p(\mathbb{R}^n) \otimes \mathbb{B}$ and $g \in L^{p'}(\mathbb{R}^n) \otimes \mathbb{B}^*$. \square

Assume now that $F \in L^p(\mathbb{R}^n) \otimes \mathbb{B}$. According to [16, Lemma 2.3] we have that

$$\|F\|_{L^p(\mathbb{R}^n, \mathbb{B})} = \sup_{\substack{g \in L^{p'}(\mathbb{R}^n) \otimes \mathbb{B}^* \\ \|g\|_{L^{p'}(\mathbb{R}^n, \mathbb{B}^*)} \leq 1}} \left| \int_{\mathbb{R}^n} \langle g(x), F(x) \rangle_{\mathbb{B}^*, \mathbb{B}} dx \right|.$$

Then, since \mathbb{B}^* is also a UMD Banach space, by using Proposition 2.1, [21, Proposition 2.2] and (20) we obtain

$$\begin{aligned}
 \|F\|_{L^p(\mathbb{R}^n, \mathbb{B})} &= \frac{2^{2\ell}}{\Gamma(2\ell)} \sup_{\substack{g \in L^{p'}(\mathbb{R}^n) \otimes \mathbb{B}^* \\ \|g\|_{L^{p'}(\mathbb{R}^n, \mathbb{B}^*)} \leq 1}} \left| \int_0^\infty \int_{\mathbb{R}^n} \langle \mathcal{G}_{\mathcal{H}, \mathbb{B}^*}^\ell(g)(x, t), \mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell(F)(x, t) \rangle_{\mathbb{B}^*, \mathbb{B}} dx \frac{dt}{t} \right| \\
 &\leq C \sup_{\substack{g \in L^{p'}(\mathbb{R}^n) \otimes \mathbb{B}^* \\ \|g\|_{L^{p'}(\mathbb{R}^n, \mathbb{B}^*)} \leq 1}} \int_{\mathbb{R}^n} \|\mathcal{G}_{\mathcal{H}, \mathbb{B}^*}^\ell(g)(x, \cdot)\|_{\gamma(H, \mathbb{B}^*)} \|\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell(F)(x, \cdot)\|_{\gamma(H, \mathbb{B})} dx \\
 &\leq C \sup_{\substack{g \in L^{p'}(\mathbb{R}^n) \otimes \mathbb{B}^* \\ \|g\|_{L^{p'}(\mathbb{R}^n, \mathbb{B}^*)} \leq 1}} \|\mathcal{G}_{\mathcal{H}, \mathbb{B}^*}^\ell(g)\|_{L^{p'}(\mathbb{R}^n, \gamma(H, \mathbb{B}^*))} \|\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell(F)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \\
 &\leq C \|\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell(F)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))}.
 \end{aligned}$$

By taking into account that $L^p(\mathbb{R}^n) \otimes \mathbb{B}$ is dense in $L^p(\mathbb{R}^n, \mathbb{B})$ and (20) we conclude that

$$\|f\|_{L^p(\mathbb{R}^n, \mathbb{B})} \leq C \|\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))},$$

where $C > 0$ does not depend on f .

2.2. (b) \Rightarrow (a) Let $\gamma \in \mathbb{R} \setminus \{0\}$. We define the imaginary power $\mathcal{H}^{i\gamma}$ of \mathcal{H} on $L^2(\mathbb{R}^n)$ as follows

$$\mathcal{H}^{i\gamma} f = \sum_{k \in \mathbb{N}^n} (2|k| + n)^{i\gamma} c_k(f) \mathfrak{h}_k, \quad f \in L^2(\mathbb{R}^n).$$

Plancherel theorem implies that $\mathcal{H}^{i\gamma}$ is bounded from $L^2(\mathbb{R}^n)$ into itself. Moreover, $\mathcal{H}^{i\gamma}$ is an spectral multiplier of Laplace transform type ([29, p. 121]) associated with the Hermite operator and $\mathcal{H}^{i\gamma}$ can be extended from $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ as a bounded operator from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$ ([2, Theorem 1.1], [3, Theorem 3]). Let \mathbb{B} be a Banach space. If $1 < p < \infty$ we can define in a natural way $\mathcal{H}^{i\gamma}$ on $L^p(\mathbb{R}^n) \otimes \mathbb{B}$ as a linear operator from $L^p(\mathbb{R}^n) \otimes \mathbb{B}$ into itself. In [2, Theorem 1.2] (see also [3, Theorem 3]) it was established that \mathbb{B} is UMD if and only if $\mathcal{H}^{i\gamma}$, $\gamma \in \mathbb{R} \setminus \{0\}$, can be extended from $L^p(\mathbb{R}^n) \otimes \mathbb{B}$ to $L^p(\mathbb{R}^n, \mathbb{B})$ as a bounded operator from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself for some (equivalently, for every) $1 < p < \infty$.

Suppose now that (b) holds. In order to see that \mathbb{B} is UMD we prove the following vector valued version of an inequality in [29, p. 63].

Proposition 2.2. *Let \mathbb{B} be a Banach space and $\gamma \in \mathbb{R} \setminus \{0\}$. There exists $C > 0$ such that, for every $f \in \text{span}\{\mathfrak{h}_k\}_{k \in \mathbb{N}^n} \otimes \mathbb{B}$,*

$$\|\mathcal{G}_{\mathcal{H}, \mathbb{B}}^1(\mathcal{H}^{i\gamma}(f))(x, \cdot)\|_{\gamma(H, \mathbb{B})} \leq C \|\mathcal{G}_{\mathcal{H}, \mathbb{B}}^2(f)(x, \cdot)\|_{\gamma(H, \mathbb{B})}, \quad x \in \mathbb{R}^n.$$

Proof. Let $f \in \text{span}\{\mathfrak{h}_k\}_{k \in \mathbb{N}^n} \otimes \mathbb{B}$. Then, $f = \sum_{k \in I} b_k \mathfrak{h}_k$, where I is a finite subset of \mathbb{N}^n and $b_k \in \mathbb{B}$, $k \in I$. We introduce the operator $U \in L(\mathbb{B})$ defined by $U(b) = -b$, $b \in \mathbb{B}$, and the operator T_γ on H , given by

$$T_\gamma(h)(t) = \frac{1}{t} \int_0^t \phi_\gamma(t-s) h(s) ds, \quad h \in H \text{ and } t > 0,$$

where $\phi_\gamma(u) = u^{-i\gamma}/\Gamma(1-i\gamma)$, $u > 0$. The operator $T_\gamma \in L(H)$ and $\|T\|_{L(H)} \leq 1/\Gamma(1-i\gamma)$. Indeed, by using Hölder's inequality and Fubini's theorem we get

$$\begin{aligned} \|T_\gamma(h)\|_H &\leq \|\phi_\gamma\|_{L^\infty(0,\infty)} \left\{ \int_0^\infty \left(\frac{1}{t} \int_0^t |h(s)| ds \right)^2 \frac{dt}{t} \right\}^{1/2} \\ &\leq \frac{1}{\Gamma(1-i\gamma)} \left\{ \int_0^\infty |h(s)|^2 \int_s^\infty \frac{dt}{t^2} ds \right\}^{1/2} = \frac{1}{\Gamma(1-i\gamma)} \|h\|_H, \quad h \in H. \end{aligned}$$

Let $x \in \mathbb{R}^n$. By considering $\mathcal{G}_{\mathcal{H},\mathbb{B}}^1(\mathcal{H}^{i\gamma}f)(x, \cdot)$ and $\mathcal{G}_{\mathcal{H},\mathbb{B}}^2(f)(x, \cdot)$ as elements of $\gamma(H, \mathbb{B})$ we have that

$$(23) \quad \mathcal{G}_{\mathcal{H},\mathbb{B}}^1(\mathcal{H}^{i\gamma}f)(x, \cdot)(h) = U\mathcal{G}_{\mathcal{H},\mathbb{B}}^2(f)(x, \cdot)T_\gamma(h), \quad h \in H.$$

In fact, for every $h \in H$ and $S \in \mathbb{B}^*$, by using well-known properties of Laplace transform, we can write

$$\begin{aligned} \langle S, U\mathcal{G}_{\mathcal{H},\mathbb{B}}^2(f)(x, \cdot)T_\gamma(h) \rangle &= -\langle S, \mathcal{G}_{\mathcal{H},\mathbb{B}}^2(f)(x, \cdot)T_\gamma(h) \rangle \\ &= -\int_0^\infty \langle S, \sum_{k \in I} b_k t^2 (2|k| + n)^2 e^{-t(2|k|+n)} \mathfrak{h}_k(x) \rangle T_\gamma(h)(t) \frac{dt}{t} \\ &= -\sum_{k \in I} \langle S, b_k \rangle (2|k| + n)^2 \mathfrak{h}_k(x) \int_0^\infty t e^{-t(2|k|+n)} T_\gamma(h)(t) dt \\ &= \left\langle S, -\sum_{k \in I} b_k (2|k| + n)^{i\gamma+1} \mathfrak{h}_k(x) \int_0^\infty e^{-t(2|k|+n)} h(t) dt \right\rangle. \end{aligned}$$

Hence,

$$U\mathcal{G}_{\mathcal{H},\mathbb{B}}^2(f)(x, \cdot)T_\gamma(h) = -\sum_{k \in I} b_k (2|k| + n)^{i\gamma+1} \mathfrak{h}_k(x) \int_0^\infty e^{-t(2|k|+n)} h(t) dt, \quad h \in H.$$

In a similar way we can see that

$$\mathcal{G}_{\mathcal{H},\mathbb{B}}^1(\mathcal{H}^{i\gamma}f)(x, \cdot)(h) = -\sum_{k \in I} b_k (2|k| + n)^{i\gamma+1} \mathfrak{h}_k(x) \int_0^\infty e^{-t(2|k|+n)} h(t) dt, \quad h \in H.$$

Thus (23) is established.

By taking into account the ideal property for the γ -radonifying operators ([35, Theorem 6.2]) we conclude that

$$\|\mathcal{G}_{\mathcal{H},\mathbb{B}}^1(\mathcal{H}^{i\gamma}f)(x, \cdot)\|_{\gamma(H,\mathbb{B})} \leq \frac{1}{\Gamma(1-i\gamma)} \|\mathcal{G}_{\mathcal{H},\mathbb{B}}^2(f)(x, \cdot)\|_{\gamma(H,\mathbb{B})}.$$

□

Let $\gamma \in \mathbb{R} \setminus \{0\}$ and $1 < p < \infty$. From (b) and Proposition 2.2 it follows, for every $f \in \text{span}\{\mathfrak{h}_k\}_{k \in \mathbb{N}^n} \otimes \mathbb{B}$,

$$\|\mathcal{H}^{i\gamma}(f)\|_{L^p(\mathbb{R}^n, \mathbb{B})} \leq C \|\mathcal{G}_{\mathcal{H},\mathbb{B}}^1(\mathcal{H}^{i\gamma}f)\|_{L^p(\mathbb{R}^n, \gamma(H,\mathbb{B}))} \leq C \|\mathcal{G}_{\mathcal{H},\mathbb{B}}^2(f)\|_{L^p(\mathbb{R}^n, \gamma(H,\mathbb{B}))} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}.$$

Since $\text{span}\{\mathfrak{h}_k\}_{k \in \mathbb{N}^n} \otimes \mathbb{B}$ is a dense subspace in $L^p(\mathbb{R}^n, \mathbb{B})$, $\mathcal{H}^{i\gamma}$ can be extended to $L^p(\mathbb{R}^n, \mathbb{B})$ as a bounded operator from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself. From [2, Theorem 1.2] ([3, Theorem 3]) we deduce that \mathbb{B} is UMD.

3. PROOF OF THEOREM 1.1 FOR THE SCHRÖDINGER OPERATOR

In this section we prove $(a) \Leftrightarrow (c)$ in Theorem 1.1. We assume that $n \geq 3$ and that the potential function V satisfies the reverse Hölder's inequality (5) where $s > n/2$. In the proof of $(c) \Rightarrow (a)$ we will use [3, Theorem 3] where UMD Banach spaces are characterized by the L^p -boundedness properties of the imaginary power $\mathcal{L}^{i\gamma}$, $\gamma \in \mathbb{R} \setminus \{0\}$, of the Schrödinger operator \mathcal{L} .

3.1. $(a) \Rightarrow (c)$ In order to show this result we can proceed as in the proof of $(a) \Rightarrow (b)$ by using in each moment the suitable property for the heat kernel $W_t^{\mathcal{L}}(x, y)$, $x, y \in \mathbb{R}^n$ and $t > 0$, of the Schrödinger semigroup.

As it is showed in the papers of Dziubański and Zienkiewicz ([10], [11] and [12]), Dziubański, Garrigós, Martínez, Torrea and Zienkiewicz ([9]) and Shen ([28]), the function ρ defined by

$$\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x, r)} V(y) dy \leq 1 \right\}, \quad x \in \mathbb{R}^n,$$

plays an important role in the develop of the harmonic analysis in the Schrödinger setting. In the special case of the Hermite operator, we can see that

$$\rho(x) \sim \begin{cases} \frac{1}{2}, & |x| \leq 1, \\ \frac{1}{1 + |x|}, & |x| \geq 1. \end{cases}$$

This function ρ is usually called “critical radius” of x , and we use it to split the operators in the local and global parts (see (13)). The main properties of the function ρ can be encountered in [28, Lemma 1.4]. We must apply repeatedly that, for every $M > 0$, $\rho(x) \sim \rho(y)$ provided that $x, y \in \mathbb{R}^n$ and $|x - y| \leq M\rho(x)$, where the equivalence constants depend only on M . Also, according to [9, Proposition 5], we can find a sequence $\{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$ such that:

- (i) $\bigcup_{k \in \mathbb{N}} B(x_k, \rho(x_k)) = \mathbb{R}^n$;
- (ii) for every $M > 0$ there exists $m \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$,

$$\text{card} \{j \in \mathbb{N} : B(x_j, M\rho(x_j)) \cap B(x_k, M\rho(x_k)) \neq \emptyset\} \leq m.$$

To complete the proof we need to use the following properties of $W_t^{\mathcal{L}}(x, y)$, $x, y \in \mathbb{R}^n$ and $t > 0$. All of them can be found, for instance in [9, Section 2] and [11, Section 2].

Lemma 3.1. *Assume that $V \in RH_s$, where $s > n/2$. Then,*

- (i) *For every $k, N \in \mathbb{N}$, there exist $C, c > 0$ for which*

$$|t^k \partial_t^k W_t^{\mathcal{L}}(x, y)| \leq C \frac{e^{-c|x-y|^2/t}}{t^{n/2}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$

- (ii) *There exists a nonnegative function $w \in S(\mathbb{R}^n)$, the Schwartz functions space, and $\delta > 0$, such that*

$$|G_{\sqrt{t}}(x - y) - W_t^{\mathcal{L}}(x, y)| \leq C \left(\frac{\sqrt{t}}{\rho(x)} \right)^{\delta} w_{\sqrt{t}}(x - y), \quad 0 < t \leq \rho(x)^2, \quad x, y \in \mathbb{R}^n.$$

(iii) If $w \in S(\mathbb{R}^n)$ there exist $\delta, \beta > 0$ such that

$$\int_{\mathbb{R}^n} G_{\sqrt{t}}(x-y)V(y)dy \leq C \begin{cases} \frac{1}{t} \left(\frac{\sqrt{t}}{\rho(x)} \right)^\delta, & 0 < t \leq \rho(x)^2, \\ \left(\frac{\sqrt{t}}{\rho(x)} \right)^{\beta+2-n}, & t > \rho(x)^2. \end{cases}$$

The polarization equality (see (21)) can be shown in the Schrödinger setting by using spectral arguments.

3.2. $\boxed{(c) \Rightarrow (a)}$ Assume that (c) holds for a certain $1 < p < \infty$.

We denote by $E_{\mathcal{L}}(d\lambda)$ the spectral measure associated to the Schrödinger operator \mathcal{L} . Then, we have that

$$W_t^{\mathcal{L}}(f) = \int_{[0,\infty)} e^{-\lambda t} E_{\mathcal{L}}(d\lambda) f, \quad f \in L^2(\mathbb{R}^n).$$

We can also write

$$W_t^{\mathcal{L}}(f)(x) = \int_{\mathbb{R}^n} W_t^{\mathcal{L}}(x, y) f(y) dy, \quad f \in L^2(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n.$$

Let $f, g \in L^2(\mathbb{R}^n)$. Then,

$$\langle \partial_t W_t^{\mathcal{L}}(f)(x), g(x) \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_t W_t^{\mathcal{L}}(x, y) f(y) dy g(x) dx = \partial_t \langle W_t^{\mathcal{L}} f(x), g(x) \rangle, \quad x \in \mathbb{R}^n \text{ and } t > 0.$$

Note that by using Lemma 3.1, (i), we can justified the derivation under the integral sign. Indeed, Lemma 3.1, (i), implies that

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\partial_t W_t^{\mathcal{L}}(x, y)| |f(y)| |g(x)| dy dx &\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{e^{-c|x-y|^2/t}}{t^{n/2+1}} |f(y)| |g(x)| dy dx \\ &\leq \frac{C}{t} \int_{\mathbb{R}^n} \sup_{s>0} \left(\int_{\mathbb{R}^n} \frac{e^{-c|x-y|^2/s}}{s^{n/2}} |f(y)| dy \right) |g(x)| dx \\ &\leq \frac{C}{t} \|g\|_{L^2(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} \left(\sup_{s>0} \int_{\mathbb{R}^n} \frac{e^{-c|x-y|^2/s}}{s^{n/2}} |f(y)| dy \right)^2 dx \right\}^{1/2} \leq \frac{C}{t}, \quad t > 0, \end{aligned}$$

because the maximal operator W_* defined by

$$W_*(F) = \sup_{s>0} |W_s(F)|, \quad F \in L^2(\mathbb{R}^n),$$

is bounded from $L^2(\mathbb{R}^n)$ into itself.

On the other hand, by defining

$$D_t W_t^{\mathcal{L}}(f) = \lim_{h \rightarrow 0} \frac{W_{t+h}^{\mathcal{L}}(f) - W_t^{\mathcal{L}}(f)}{h}, \quad \text{on } L^2(\mathbb{R}^n),$$

we have that

$$D_t W_t^{\mathcal{L}}(f) = - \int_{[0,\infty)} \lambda e^{-\lambda t} E_{\mathcal{L}}(d\lambda) f, \quad t > 0.$$

Hence, we conclude that, for every $t > 0$,

$$D_t W_t^{\mathcal{L}}(f)(x) = \int_{\mathbb{R}^n} \partial_t W_t^{\mathcal{L}}(x, y) f(y) dy, \quad \text{a.e. } x \in \mathbb{R}^n.$$

Then, for every $f \in L^2(\mathbb{R}^n) \otimes \mathbb{B}$, $\ell = 1, 2$, and $t > 0$,

$$\mathcal{G}_{\mathcal{L}, \mathbb{B}}^{\ell}(f)(\cdot, t) = \int_{[0,\infty)} (-\lambda t)^{\ell} e^{-\lambda t} E_{\mathcal{L}}(d\lambda) f,$$

where the right hand side has the obvious meaning.

Let $\gamma \in \mathbb{R} \setminus \{0\}$. The imaginary power $\mathcal{L}^{i\gamma}$ of the operator \mathcal{L} is defined by

$$\mathcal{L}^{i\gamma}(f) = \int_{[0,\infty)} \lambda^{i\gamma} E_{\mathcal{L}}(d\lambda) f, \quad f \in L^2(\mathbb{R}^n),$$

and we extend $\mathcal{L}^{i\gamma}$ to $L^2(\mathbb{R}^n) \otimes \mathbb{B}$ in the natural way.

It is clear that, for every $t > 0$,

$$\mathcal{G}_{\mathcal{L},\mathbb{B}}^1(\mathcal{L}^{i\gamma}f)(\cdot, t) = - \int_{[0,\infty)} t \lambda^{1+i\gamma} e^{-\lambda t} E_{\mathcal{L}}(d\lambda) f, \quad f \in L^2(\mathbb{R}^n) \otimes \mathbb{B}.$$

In the following we establish the analogous property of Proposition 2.2 but in the Schrödinger setting.

Proposition 3.1. *Let \mathbb{B} be a Banach space and $\gamma \in \mathbb{R} \setminus \{0\}$. There exists $C > 0$ such that, for every $f \in S(\mathbb{R}^n) \otimes \mathbb{B}$,*

$$\|\mathcal{G}_{\mathcal{L},\mathbb{B}}^1(\mathcal{L}^{i\gamma}(f))(x, \cdot)\|_{\gamma(H,\mathbb{B})} \leq C \|\mathcal{G}_{\mathcal{L},\mathbb{B}}^2(f)(x, \cdot)\|_{\gamma(H,\mathbb{B})}, \quad a.e. \ x \in \mathbb{R}^n.$$

Proof. Let $h \in L^2((0, \infty), dt/t)$ such that $\text{supp } h \subset (a, b)$, $0 < a < b < \infty$, and let $f, g \in L^2(\mathbb{R}^n)$. According to Lemma 3.1, (i), we get as above

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^n} |\mathcal{G}_{\mathcal{L},\mathbb{B}}^1(\mathcal{L}^{i\gamma}f)(x, t)g(x)| dx |h(t)| \frac{dt}{t} \\ & \leq C \int_0^\infty \int_{\mathbb{R}^n} \sup_{s>0} \left(\int_{\mathbb{R}^n} \frac{e^{-c|x-y|^2/t}}{s^{n/2}} |\mathcal{L}^{i\gamma}f(y)| dy \right) |g(x)| dx |h(t)| \frac{dt}{t} \\ & \leq C \|g\|_{L^2(\mathbb{R}^n)} \|W_*(\mathcal{L}^{i\gamma}f)\|_{L^2(\mathbb{R}^n)} \int_a^b |h(t)| \frac{dt}{t} < \infty. \end{aligned}$$

We can write

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_0^\infty \mathcal{G}_{\mathcal{L},\mathbb{B}}^1(\mathcal{L}^{i\gamma}f)(x, t) h(t) \frac{dt}{t} g(x) dx \\ & = \int_0^\infty h(t) \int_{\mathbb{R}^n} \mathcal{G}_{\mathcal{L},\mathbb{B}}^1(\mathcal{L}^{i\gamma}f)(x, t) g(x) dx \frac{dt}{t} \\ & = - \int_0^\infty h(t) \int_{\mathbb{R}^n} \left(\int_{[0,\infty)} t \lambda^{1+i\gamma} e^{-\lambda t} E_{\mathcal{L}}(d\lambda) f \right) (x) g(x) dx \frac{dt}{t} \\ & = - \int_0^\infty \int_{[0,\infty)} \lambda^{1+i\gamma} e^{-\lambda t} d\mu_{f,g}(\lambda) h(t) dt, \end{aligned}$$

where $\mu_{f,g}$ represents the complex measure defined by

$$\mu_{f,g}(A) = \langle E_{\mathcal{L}}(A)f, g \rangle,$$

for every Borel set A in $[0, \infty)$. If $|\mu_{f,g}|$ denotes the total variation measure of $\mu_{f,g}$, then

$$\int_0^\infty \int_{[0,\infty)} |\lambda^{1+i\gamma}| e^{-\lambda t} d|\mu_{f,g}|(\lambda) |h(t)| dt \leq C |\mu_{f,g}|([0, \infty)) \int_0^\infty |h(t)| \frac{dt}{t} < \infty.$$

Hence, we have that

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_0^\infty \mathcal{G}_{\mathcal{L},\mathbb{B}}^1(\mathcal{L}^{i\gamma}f)(x, t) h(t) \frac{dt}{t} g(x) dx \\ & = - \int_{[0,\infty)} \lambda^{1+i\gamma} \int_0^\infty e^{-\lambda t} h(t) dt d\mu_{f,g}(\lambda) \\ & = - \int_{[0,\infty)} \lambda^2 \int_0^\infty t^2 e^{-\lambda t} T_\gamma(h)(t) \frac{dt}{t} d\mu_{f,g}(\lambda), \end{aligned}$$

where

$$T_\gamma(h)(t) = \frac{1}{t} \int_0^t \phi_\gamma(t-s)h(s)ds, \quad t \in (0, \infty),$$

and $\phi_\gamma(u) = u^{-i\gamma}/\Gamma(1-i\gamma)$, $u \in (0, \infty)$. Since

$$\int_{[0, \infty)} \int_0^\infty (\lambda t)^2 e^{-\lambda t} |T_\gamma(h)(t)| \frac{dt}{t} d|\mu_{f,g}|(\lambda) < \infty,$$

we can write

$$\begin{aligned} \int_{\mathbb{R}^n} \int_0^\infty \mathcal{G}_{\mathcal{L}, \mathbb{B}}^1(\mathcal{L}^{i\gamma} f)(x, t) h(t) \frac{dt}{t} g(x) dx &= - \int_0^\infty T_\gamma(h)(t) \int_{[0, \infty)} (\lambda t)^2 e^{-\lambda t} d\mu_{f,g}(\lambda) \frac{dt}{t} \\ &= - \int_0^\infty T_\gamma(h)(t) \int_{\mathbb{R}^n} g(x) \mathcal{G}_{\mathcal{L}, \mathbb{C}}^2(f)(x, t) dx \frac{dt}{t} \\ &= - \int_{\mathbb{R}^n} g(x) \int_0^\infty T_\gamma(h)(t) \mathcal{G}_{\mathcal{L}, \mathbb{C}}^2(f)(x, t) \frac{dt}{t} dx. \end{aligned}$$

The last interchange is justified because

$$\int_0^\infty \int_{\mathbb{R}^n} |g(x)| |\mathcal{G}_{\mathcal{L}, \mathbb{C}}^2(f)(x, t)| dx T_\gamma(h)(t) \frac{dt}{t} \leq C \|h\|_H \int_a^\infty \int_{\mathbb{R}^n} |g(x)| W_*(|f|)(x) dx \frac{dt}{t^2} < \infty.$$

We have taken into account that, since $\text{supp } h \subset (a, b)$, it follows that $T_\gamma(h)(t) = 0$, when $t \in (0, a)$.

We conclude that

$$\int_0^\infty \mathcal{G}_{\mathcal{L}, \mathbb{C}}^1(\mathcal{L}^{i\gamma} f)(x, t) h(t) \frac{dt}{t} = - \int_0^\infty T_\gamma(h)(t) \mathcal{G}_{\mathcal{L}, \mathbb{C}}^2(f)(x, t) \frac{dt}{t}, \quad \text{a.e. } x \in \mathbb{R}^n.$$

It is well-known that the space $C_c(0, \infty)$ of continuous functions with compact support is dense in H . Moreover, since H is separable, there exists a numerable set $\mathcal{A} \subset C_c(0, \infty)$ that is dense in H .

We define $\mathbf{N} \subset \mathbb{R}^n$ consisting on those $x \in \mathbb{R}^n$ for which

$$\int_0^\infty \mathcal{G}_{\mathcal{L}, \mathbb{C}}^1(\mathcal{L}^{i\gamma} f)(x, t) h(t) \frac{dt}{t} = - \int_0^\infty T_\gamma(h)(t) \mathcal{G}_{\mathcal{L}, \mathbb{C}}^2(f)(x, t) \frac{dt}{t}, \quad h \in \mathcal{A}.$$

We have that $|\mathbb{R}^n \setminus \mathbf{N}| = 0$. then, for every $h \in H$,

$$\int_0^\infty \mathcal{G}_{\mathcal{L}, \mathbb{C}}^1(\mathcal{L}^{i\gamma} f)(x, t) h(t) \frac{dt}{t} = - \int_0^\infty T_\gamma(h)(t) \mathcal{G}_{\mathcal{L}, \mathbb{C}}^2(f)(x, t) \frac{dt}{t}, \quad x \in \mathbf{N}.$$

Hence, if $f \in L^2(\mathbb{R}^n) \otimes \mathbb{B}$, there exists $\Omega \subset \mathbb{R}^n$ such that $|\mathbb{R}^n \setminus \mathbf{N}| = 0$ and

$$\int_0^\infty \mathcal{G}_{\mathcal{L}, \mathbb{B}}^1(\mathcal{L}^{i\gamma} f)(x, t) h(t) \frac{dt}{t} = - \int_0^\infty T_\gamma(h)(t) \mathcal{G}_{\mathcal{L}, \mathbb{B}}^2(f)(x, t) \frac{dt}{t}, \quad x \in \Omega,$$

for every $h \in H$. By defining $Ub = -b$, $b \in \mathbb{B}$, we have that, as elements of $\gamma(H, \mathbb{B})$,

$$\mathcal{G}_{\mathcal{L}, \mathbb{B}}^1(\mathcal{L}^{i\gamma} f)(x, \cdot) = U \mathcal{G}_{\mathcal{L}, \mathbb{B}}^2(f)(x, \cdot) T_\gamma, \quad \text{a.e. } x \in \mathbb{R}^n,$$

for every $f \in S(\mathbb{R}^n) \otimes \mathbb{B}$.

By taking into account the ideal property of $\gamma(H, \mathbb{B})$ ([35, Theorem 6.2]), and that the operators U and T_γ are bounded in \mathbb{B} and H , respectively, we conclude the proof of this proposition. \square

Finally, from the equivalences in (c) and Proposition 3.1, we have that, for every $f \in S(\mathbb{R}^n) \otimes \mathbb{B}$,

$$\|\mathcal{L}^{i\gamma}\|_{L^p(\mathbb{R}^n, \mathbb{B})} \leq C \|\mathcal{G}_{\mathcal{L}, \mathbb{B}}^1(\mathcal{L}^{i\gamma} f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \leq C \|\mathcal{G}_{\mathcal{L}, \mathbb{B}}^2(f)\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}.$$

Since $S(\mathbb{R}^n) \otimes \mathbb{B}$ is dense in $L^p(\mathbb{R}^n, \mathbb{B})$, we have proved that the operator $\mathcal{L}^{i\gamma}$ can be extended to $L^p(\mathbb{R}^n, \mathbb{B})$ as a bounded operator from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself. Then, according to [3, Theorem 3], \mathbb{B} is UMD.

4. PROOF OF THEOREM 1.1 FOR LAGUERRE OPERATORS

In this section we prove the equivalence (a) \Leftrightarrow (d) in Theorem 1.1.

Suppose that \mathbb{B} is a UMD Banach space. Let $\ell = 1, 2$ and $1 < p < \infty$.

We are going to see that

$$(24) \quad \|\mathcal{G}_{\mathcal{L}_\alpha, \mathbb{B}}^\ell(f)\|_{L^p((0, \infty), \gamma(H, \mathbb{B}))} \leq C \|f\|_{L^p((0, \infty), \mathbb{B})}, \quad f \in L^p((0, \infty), \mathbb{B}).$$

In order to show (24) we take advantage from (a) \Rightarrow (b) after connecting $\mathcal{G}_{\mathcal{L}_\alpha, \mathbb{B}}^\ell$ and $\mathcal{G}_{\mathcal{H}, \mathbb{B}}^\ell$ in a suitable way.

Note firstly that

$$(25) \quad W_t^{\mathcal{L}_\alpha}(x, y) = W_t^{\mathcal{H}/2}(x, y) g_\alpha \left(\frac{2xye^{-t}}{1 - e^{-2t}} \right), \quad x, y, t \in (0, \infty),$$

where $W_t^{\mathcal{H}/2}(x, y)$ denotes the heat kernel associated with the operator $\mathcal{H}/2$ in dimension one, that is, for each $x, y \in \mathbb{R}$ and $t > 0$,

$$W_t^{\mathcal{H}/2}(x, y) = \left(\frac{e^{-t}}{\pi(1 - e^{-2t})} \right)^{1/2} \exp \left[-\frac{1}{4} \left((x - y)^2 \frac{1 + e^{-t}}{1 - e^{-t}} + (x + y)^2 \frac{1 - e^{-t}}{1 + e^{-t}} \right) \right],$$

and g_α is defined by

$$g_\alpha(z) = \sqrt{2\pi z} e^{-z} I_\alpha(z), \quad z \in (0, \infty).$$

To make the reading of the following lines easier, from now on we consider $\xi = \xi(x, y, t) = \frac{2xye^{-t}}{1 - e^{-2t}}$, $x, y, t \in (0, \infty)$.

We have, for every $x, y, t \in (0, \infty)$,

$$(26) \quad \partial_t W_t^{\mathcal{L}_\alpha}(x, y) = \partial_t W_t^{\mathcal{H}/2}(x, y) g_\alpha(\xi) - W_t^{\mathcal{H}/2}(x, y) \left(\frac{d}{dz} g_\alpha(z) \right) \Big|_{z=\xi} \frac{\xi(1 + e^{-2t})}{1 - e^{-2t}},$$

and

$$(27) \quad \begin{aligned} \partial_t^2 W_t^{\mathcal{L}_\alpha}(x, y) &= \partial_t^2 W_t^{\mathcal{H}/2}(x, y) g_\alpha(\xi) - 2\partial_t W_t^{\mathcal{H}/2}(x, y) \left(\frac{d}{dz} g_\alpha(z) \right) \Big|_{z=\xi} \frac{\xi(1 + e^{-2t})}{1 - e^{-2t}} \\ &\quad + W_t^{\mathcal{H}/2}(x, y) \left\{ \left(\frac{d^2}{dz^2} g_\alpha(z) \right) \Big|_{z=\xi} \frac{\xi^2(1 + e^{-2t})^2}{(1 - e^{-2t})^2} + \left(\frac{d}{dz} g_\alpha(z) \right) \Big|_{z=\xi} \frac{\xi(1 + 6e^{-2t} + e^{-4t})}{(1 - e^{-2t})^2} \right\}. \end{aligned}$$

By taking into account that $\frac{d}{dz}(z^{-\alpha} I_\alpha(z)) = z^{-\alpha} I_{\alpha+1}(z)$, $z \in (0, \infty)$ ([24, p. 110]), we get

$$(28) \quad \frac{d}{dz} g_\alpha(z) = -g_\alpha(z) + \frac{2\alpha + 1}{2z} g_\alpha(z) + g_{\alpha+1}(z), \quad z \in (0, \infty),$$

and

$$(29) \quad \frac{d^2}{dz^2} g_\alpha(z) = \left(1 - \frac{2\alpha + 1}{z} + \frac{4\alpha^2 - 1}{4z^2} \right) g_\alpha(z) + 2 \left(\frac{\alpha - 1}{z} - 1 \right) g_{\alpha+1}(z) + g_{\alpha+2}(z), \quad z \in (0, \infty).$$

Since $I_\alpha(z) \sim z^\alpha$, as $z \rightarrow 0^+$ ([24, p. 108]), we deduce from (28) and (29) that, for $k = 0, 1, 2$,

$$(30) \quad \left| z^k \frac{d^k}{dz^k} g_\alpha(z) \right| \leq C, \quad z \in (0, 1).$$

On the other hand, according to [24, p. 123], for every $m \in \mathbb{N}$,

$$(31) \quad g_\alpha(z) = \sum_{r=0}^m \frac{(-1)^r [\alpha, r]}{(2z)^r} + \mathcal{O} \left(\frac{1}{z^{m+1}} \right), \quad z \in (0, \infty),$$

where $[\alpha, 0] = 1$ and

$$[\alpha, r] = \frac{(4\alpha^2 - 1)(4\alpha^2 - 3^2) \cdots (4\alpha^2 - (2r - 1)^2)}{2^{2r} \Gamma(r + 1)}, \quad r \in \mathbb{N}, \quad r \geq 1.$$

Then, for $k = 1, 2$,

$$(32) \quad \left| z^k \frac{d^k}{dz^k} g_\alpha(z) \right| \leq \frac{C}{z}, \quad z \in (0, \infty).$$

Indeed, by using property (31) in (28) and (29) we get

$$\frac{d}{dz} g_\alpha(z) = \frac{[\alpha, 1] - [\alpha + 1, 1] + 2\alpha + 1}{2z} + \mathcal{O}\left(\frac{1}{z^2}\right) = \mathcal{O}\left(\frac{1}{z^2}\right), \quad z \in (0, \infty).$$

and

$$\begin{aligned} \frac{d^2}{dz^2} g_\alpha(z) &= \left(1 - \frac{[\alpha, 1]}{2} + [\alpha + 1, 1] - \frac{[\alpha + 2, 1]}{2} \right) \frac{1}{z} \\ &\quad + \left(\frac{[\alpha, 2]}{4} + (2\alpha + 3) \frac{[\alpha, 1]}{2} - (\alpha + 1)[\alpha + 1, 1] - \frac{[\alpha + 1, 2]}{2} + \frac{[\alpha + 2, 2]}{4} \right) \frac{1}{z^2} \\ &\quad + \mathcal{O}\left(\frac{1}{z^3}\right) = \mathcal{O}\left(\frac{1}{z^3}\right), \quad z \in (0, \infty). \end{aligned}$$

Then, (32) holds for $k = 1, 2$.

From (26) and (27) by using (12) (note that estimate (12) also holds for $\mathcal{H}/2$ instead of \mathcal{H}) we obtain, for every $x, y, t \in (0, \infty)$,

$$\begin{aligned} (33) \quad \left| t^\ell \partial_t^\ell W_t^{\mathcal{L}_\alpha}(x, y) \right| &\leq C \left\{ \left| t^\ell \partial_t^\ell W_t^{\mathcal{H}/2}(x, y) \right| |g_\alpha(\xi)| + \left| t^\ell \partial_t^{\ell-1} W_t^{\mathcal{H}/2}(x, y) \right| \left| \left(\frac{d}{dz} g_\alpha(z) \right) \Big|_{z=\xi} \right| \frac{\xi}{1 - e^{-2t}} \right. \\ &\quad \left. + (\ell - 1) t^2 W_t^{\mathcal{H}/2}(x, y) \left[\left| \left(\frac{d^2}{dz^2} g_\alpha(z) \right) \Big|_{z=\xi} \right| \frac{\xi^2}{(1 - e^{-2t})^2} + \left| \left(\frac{d}{dz} g_\alpha(z) \right) \Big|_{z=\xi} \right| \frac{\xi}{(1 - e^{-2t})^2} \right] \right\} \\ &\leq C \frac{e^{-t/3} e^{-c|x-y|^2/t}}{\sqrt{t}} \left\{ |g_\alpha(\xi)| + \frac{t\xi}{1 - e^{-2t}} \left| \left(\frac{d}{dz} g_\alpha(z) \right) \Big|_{z=\xi} \right| + (\ell - 1) \frac{(t\xi)^2}{(1 - e^{-2t})^2} \left| \left(\frac{d^2}{dz^2} g_\alpha(z) \right) \Big|_{z=\xi} \right| \right\}. \end{aligned}$$

Now, (30) implies that

$$(34) \quad \left| t^\ell \partial_t^\ell W_t^{\mathcal{L}_\alpha}(x, y) \right| \leq C \frac{e^{-c|x-y|^2/t}}{\sqrt{t}}, \quad x, y, t \in (0, \infty) \text{ and } \xi \leq 1.$$

We also observe that

$$\begin{aligned} \exp \left[-c \left(\frac{1 + e^{-t}}{1 - e^{-t}} |x - y|^2 + \frac{1 - e^{-t}}{1 + e^{-t}} |x + y|^2 \right) \right] &= \exp \left[-2c \left(\frac{1 + e^{-2t}}{1 - e^{-2t}} (x^2 + y^2) - 8 \frac{e^{-t}}{1 - e^{-2t}} xy \right) \right] \\ &\leq \exp \left[-c \frac{1 + e^{-2t}}{1 - e^{-2t}} (x^2 + y^2) \right], \quad x, y, t \in (0, \infty) \text{ and } \xi \leq 1. \end{aligned}$$

Then we get

$$(35) \quad \left| t^\ell \partial_t^\ell W_t^{\mathcal{H}/2}(x, y) \right| \leq C \frac{e^{-c(x^2+y^2)/t}}{\sqrt{t}}, \quad x, y, t \in (0, \infty), \quad \xi \leq 1,$$

and

$$(36) \quad \left| t^\ell \partial_t^\ell W_t^{\mathcal{L}_\alpha}(x, y) \right| \leq C \frac{e^{-c(x^2+y^2)/t}}{\sqrt{t}}, \quad x, y, t \in (0, \infty) \text{ and } \xi \leq 1.$$

On the other hand, from (32) and (33) it follows that,

$$(37) \quad \left| t^\ell \partial_t^\ell W_t^{\mathcal{L}_\alpha}(x, y) \right| \leq C \frac{xy e^{-c|x-y|^2/t}}{t^{3/2}}, \quad x, y, t \in (0, \infty) \text{ and } \xi \geq 1.$$

Moreover, by taking into account (12), (25), (31) and (32), we obtain that

$$\begin{aligned}
 & \left| t^\ell \partial_t^\ell [W_t^{\mathcal{L}_\alpha}(x, y) - W_t^{\mathcal{H}/2}(x, y)] \right| \\
 & \leq \left| t^\ell \partial_t^\ell [W_t^{\mathcal{H}/2}(x, y)](g_\alpha(\xi) - 1) \right| + \left| t^\ell \partial_t^{\ell-1} W_t^{\mathcal{H}/2}(x, y) \right| \left| \left(\frac{d}{dz} g_\alpha(z) \right) \Big|_{z=\xi} \right| \frac{\xi}{1 - e^{-2t}} \\
 & \quad + (\ell - 1) t^2 W_t^{\mathcal{H}/2}(x, y) \left\{ \left| \left(\frac{d^2}{dz^2} g_\alpha(z) \right) \Big|_{z=\xi} \right| \frac{\xi^2}{(1 - e^{-2t})^2} + \left| \left(\frac{d}{dz} g_\alpha(z) \right) \Big|_{z=\xi} \right| \frac{\xi}{(1 - e^{-2t})^2} \right\} \\
 & \leq C \frac{e^{-t/3} e^{-c|x-y|^2/t}}{\xi \sqrt{t}}, \quad x, y, t \in (0, \infty).
 \end{aligned}$$

Hence, we get

$$(38) \quad \left| t^\ell \partial_t^\ell [W_t^{\mathcal{L}_\alpha}(x, y) - W_t^{\mathcal{H}/2}(x, y)] \right| \leq C \frac{e^{-t/3} e^{-c|x-y|^2/t}}{\xi^{1/4} \sqrt{t}} \leq C \frac{e^{-c|x-y|^2/t}}{(xyt)^{1/4}}, \quad x, y, t \in (0, \infty) \text{ and } \xi \geq 1.$$

Let now $f \in L^p((0, \infty), \mathbb{B})$. Let us denote by \tilde{f} the extension of f to \mathbb{R} which satisfies $\tilde{f}(x) = 0$, $x \leq 0$. By defining $\mathcal{G}_{\mathcal{H}/2, \mathbb{B}}^\ell(\tilde{f})$ in the obvious way, we have that

$$\begin{aligned}
 & \|\mathcal{G}_{\mathcal{L}_\alpha, \mathbb{B}}^\ell(f)(x, \cdot) - \mathcal{G}_{\mathcal{H}/2, \mathbb{B}}^\ell(\tilde{f})(x, \cdot)\|_{L^2((0, \infty), \frac{dt}{t}; \mathbb{B})} \\
 & \leq \int_{(0, x/2) \cup (2x, \infty)} \|f(y)\|_{\mathbb{B}} \left(\left\| t^\ell \partial_t^\ell W_t^{\mathcal{L}_\alpha}(x, y) \right\|_H + \left\| t^\ell \partial_t^\ell W_t^{\mathcal{H}/2}(x, y) \right\|_H \right) dy \\
 & \quad + \int_{x/2}^{2x} \|f(y)\|_{\mathbb{B}} \left\| t^\ell \partial_t^\ell [W_t^{\mathcal{L}_\alpha}(x, y) - W_t^{\mathcal{H}/2}(x, y)] \right\|_H dy \\
 (39) \quad & = T_1(f)(x) + T_2(f)(x), \quad x \in (0, \infty).
 \end{aligned}$$

By using (12), (34) and (37) we obtain, when $x \in (0, \infty)$ and $y \in (0, x/2) \cup (2x, \infty)$,

$$\begin{aligned}
 & \left\| t^\ell \partial_t^\ell W_t^{\mathcal{L}_\alpha}(x, y) \right\|_H + \left\| t^\ell \partial_t^\ell W_t^{\mathcal{H}/2}(x, y) \right\|_H \leq C \left(1 + \frac{xy}{|x-y|^2} \right) \left(\int_0^\infty \frac{e^{-c|x-y|^2/t}}{t^2} dt \right)^{1/2} \\
 & \leq C \left(\int_0^\infty \frac{e^{-c|x-y|^2/t}}{t^2} dt \right)^{1/2} \leq C \frac{1}{|x-y|} \leq C \begin{cases} \frac{1}{x}, & 0 < y < \frac{x}{2} \\ \frac{1}{y}, & 0 < 2x < y \end{cases},
 \end{aligned}$$

because $|x-y| \sim x$, when $y \in (0, x/2)$ and $|x-y| \sim y$, when $y \in (2x, \infty)$.

Hence,

$$(40) \quad T_1(f)(x) \leq C [H_0(\|f\|_{\mathbb{B}})](x) + H_\infty(\|f\|_{\mathbb{B}})(x) < \infty, \quad x \in (0, \infty),$$

where H_0 and H_∞ represents the classical Hardy operators given by

$$H_0(g)(x) = \frac{1}{x} \int_0^x g(y) dy \quad \text{and} \quad H_\infty(g)(x) = \int_x^\infty \frac{g(y)}{y} dy, \quad x \in (0, \infty).$$

On the other hand, by taking into account (35), (36) and (38), we can write,

$$\begin{aligned}
 & \left\| t^\ell \partial_t^\ell [W_t^{\mathcal{L}_\alpha}(x, y) - W_t^{\mathcal{H}/2}(x, y)] \right\|_H \leq \left\{ \left(\int_{0, \xi \leq 1}^\infty + \int_{0, \xi \geq 1}^\infty \right) \left| t^\ell \partial_t^\ell [W_t^{\mathcal{L}_\alpha}(x, y) - W_t^{\mathcal{H}/2}(x, y)] \right|^2 \frac{dt}{t} \right\}^{1/2} \\
 & \leq C \left(\int_0^\infty \frac{e^{-c(x^2+y^2)/t}}{t^2} dt + \frac{1}{(xy)^{1/2}} \int_0^\infty \frac{e^{-c|x-y|^2/t}}{t^{3/2}} dt \right)^{1/2} \\
 & \leq C \left(\frac{1}{\sqrt{x^2+y^2}} + \frac{1}{(xy)^{1/4} \sqrt{|x-y|}} \right) \leq \frac{C}{y} \left(1 + \sqrt{\frac{y}{|x-y|}} \right), \quad 0 < \frac{x}{2} < y < 2x.
 \end{aligned}$$

Hence,

$$(41) \quad T_2(f)(x) \leq C\mathcal{N}(\|f\|_{\mathbb{B}})(x) < \infty, \quad x \in (0, \infty),$$

where

$$\mathcal{N}(g)(x) = \int_{x/2}^{2x} \frac{1}{y} \left(1 + \sqrt{\frac{y}{|x-y|}}\right) g(y) dy, \quad x \in (0, \infty).$$

From (39), (40) and (41) we conclude that, for every $x \in (0, \infty)$,

$$\|\mathcal{G}_{\mathcal{L}_\alpha, \mathbb{B}}^\ell(f)(x, \cdot) - \mathcal{G}_{\mathcal{H}/2, \mathbb{B}}^\ell(\tilde{f})(x, \cdot)\|_{\gamma(H, \mathbb{B})} \leq C [H_0(\|f\|_{\mathbb{B}})(x) + H_\infty(\|f\|_{\mathbb{B}})(x) + \mathcal{N}(\|f\|_{\mathbb{B}})(x)] < \infty.$$

It is well-known that the Hardy operators H_0 and H_∞ are bounded from $L^p(0, \infty)$ into itself (see [18, p. 244, (9.9.1) and (9.9.2)]). Moreover, Jensen's inequality allows us to show that the operator \mathcal{N} is also bounded from $L^p(0, \infty)$ into itself. Hence,

$$\|\mathcal{G}_{\mathcal{L}_\alpha, \mathbb{B}}^\ell(f) - \mathcal{G}_{\mathcal{H}/2, \mathbb{B}}^\ell(\tilde{f})\|_{L^p((0, \infty), \gamma(H, \mathbb{B}))} \leq C\|f\|_{L^p((0, \infty), \mathbb{B})}.$$

Since, as it was seen in Section 2, (b) holds provided that \mathbb{B} is UMD, it follows that

$$\|\mathcal{G}_{\mathcal{L}_\alpha, \mathbb{B}}^\ell(f)\|_{L^p((0, \infty), \gamma(H, \mathbb{B}))} \leq C\|f\|_{L^p((0, \infty), \mathbb{B})}.$$

Note that we can obtain results analogous to Propositions 2.1 and 2.2 for the operator \mathcal{L}_α instead of \mathcal{H} . The remainder of the proof of (a) \Rightarrow (d) and the proof of (d) \Rightarrow (a) can be made by proceeding as in the proof of the corresponding properties in Section 2.

REFERENCES

- [1] J. J. BETANCOR, A. J. CASTRO, J. CURBELO, J. C. FARIÑA, AND L. RODRÍGUEZ-MESA, *Square functions in the Hermite setting for functions with values in UMD spaces*. Preprint 2012 ([arXiv:1203.1480v1](#)).
- [2] J. J. BETANCOR, A. J. CASTRO, J. CURBELO, AND L. RODRÍGUEZ-MESA, *Characterization of UMD Banach spaces by imaginary powers of Hermite and Laguerre operators*. To appear in *Complex Anal. Oper. Theory* (DOI: 10.1007/s11785-011-0203-9).
- [3] J. J. BETANCOR, R. CRESCIMBENI, J. C. FARIÑA, AND L. RODRÍGUEZ-MESA, *Multipliers and imaginary powers of the Schrödinger operators characterizing UMD Banach spaces*. Preprint 2011 ([arXiv:1109.0429v1](#)).
- [4] J. J. BETANCOR, R. CRESCIMBENI, J. C. FARIÑA, P. R. STINGA, AND J. L. TORREA, *A T1 criterion for Hermite-Calderón-Zygmund operators on the $BMO_H(\mathbb{R}^n)$ space and applications*. To appear in *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) ([arXiv:1006.0416v2](#)).
- [5] J. J. BETANCOR, S. M. MOLINA, AND L. RODRÍGUEZ-MESA, *Area Littlewood-Paley functions associated with Hermite and Laguerre operators*, *Potential Anal.*, 34 (2011), pp. 345–369.
- [6] J. BOURGAIN, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, *Ark. Mat.*, 21 (1983), pp. 163–168.
- [7] D. L. BURKHOLDER, *Martingales and Fourier analysis in Banach spaces*, in *Probability and Analysis* (Varenna, 1985), vol. 1206 of *Lecture Notes in Math.*, Springer, Berlin, 1986, pp. 61–108.
- [8] C.-P. CHEN, D. FAN, AND S. SATO, *DeLeeuw's theorem on Littlewood-Paley functions*, *Nagoya Math. J.*, 166 (2001), pp. 23–42.
- [9] J. DZIUBAŃSKI, G. GARRIGÓS, T. MARTÍNEZ, J. L. TORREA, AND J. ZIENKIEWICZ, *BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, *Math. Z.*, 249 (2005), pp. 329–356.
- [10] J. DZIUBAŃSKI AND J. ZIENKIEWICZ, *Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, *Rev. Mat. Iberoamericana*, 15 (1999), pp. 279–296.
- [11] ———, *H^p spaces for Schrödinger operators*, in *Fourier analysis and related topics* (Bedlewo, 2000), vol. 56 of *Banach Center Publ.*, Polish Acad. Sci., Warsaw, 2002, pp. 45–53.
- [12] ———, *H^p spaces associated with Schrödinger operators with potentials from reverse Hölder inequality*, *Colloq. Math.*, 98 (2003), pp. 5–38.

- [13] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. G. TRICOMI, *Tables of integral transforms. Vol. I*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.
- [14] D. FAN AND S. SATO, *Remarks on Littlewood-Paley functions and singular integrals*, J. Math. Soc. Japan, 54 (2002), pp. 565–585.
- [15] G. B. FOLLAND, *Real analysis. Modern techniques and their applications*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, second ed., 1999.
- [16] L. GRAFAKOS, L. LIU, AND D. YANG, *Vector-valued singular integrals and maximal functions on spaces of homogeneous type*, Math. Scand., 104 (2009), pp. 296–310.
- [17] E. HARBOURE, J. L. TORREA, AND B. VIVIANI, *Vector-valued extensions of operators related to the Ornstein-Uhlenbeck semigroup*, J. Anal. Math., 91 (2003), pp. 1–29.
- [18] G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, Cambridge, at the University Press, 1934.
- [19] T. HYTÖNEN, *Littlewood-Paley-Stein theory for semigroups in UMD spaces*, Rev. Mat. Iberoam., 23 (2007), pp. 973–1009.
- [20] T. HYTÖNEN, J. VAN NEERVEN, AND P. PORTAL, *Conical square function estimates in UMD Banach spaces and applications to H^∞ -functional calculi*, J. Anal. Math., 106 (2008), pp. 317–351.
- [21] T. HYTÖNEN AND L. WEIS, *The Banach space-valued BMO, Carleson's condition, and paraproducts*, J. Fourier Anal. Appl., 16 (2010), pp. 495–513.
- [22] C. KAISER, *Wavelet transforms for functions with values in Lebesgue spaces*, in Wavelets XI, vol. 5914 of Proc. of SPIE, Bellingham, WA, 2005.
- [23] C. KAISER AND L. WEIS, *Wavelet transform for functions with values in UMD spaces*, Studia Math., 186 (2008), pp. 101–126.
- [24] N. N. LEBEDEV, *Special functions and their applications*, Dover Publications Inc., New York, 1972.
- [25] T. MARTÍNEZ, J. L. TORREA, AND Q. XU, *Vector-valued Littlewood-Paley-Stein theory for semigroups*, Adv. Math., 203 (2006), pp. 430–475.
- [26] S. MEDA, *A general multiplier theorem*, Proc. Amer. Math. Soc., 110 (1990), pp. 2201–2212.
- [27] J. L. RUBIO DE FRANCIA, *Martingale and integral transforms of Banach space valued functions*, in Probability and Banach spaces (Zaragoza, 1985), vol. 1221 of Lecture Notes in Math., Springer, Berlin, 1986, pp. 195–222.
- [28] Z. W. SHEN, *L^p estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble), 45 (1995), pp. 513–546.
- [29] E. M. STEIN, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J., 1970.
- [30] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, vol. 43 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1993.
- [31] G. SZEGŐ, *Orthogonal polynomials*, American Mathematical Society, Providence, R.I., fourth ed., 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.
- [32] L. TANG, *Weighted norm inequalities, spectral multipliers and Littlewood-Paley operators in the Schrödinger settings*. Preprint 2012 (arXiv:1203.0375v1).
- [33] S. THANGAVELU, *Lectures on Hermite and Laguerre expansions*, vol. 42 of Mathematical Notes, Princeton University Press, Princeton, NJ, 1993.
- [34] J. L. TORREA AND C. ZHANG, *Fractional vector-valued Littlewood-Paley-Stein theory for semigroups*. Preprint 2011 (arXiv:1105.6022v3).
- [35] J. VAN NEERVEN, *γ -radonifying operators—a survey*, in The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis, vol. 44 of Proc. Centre Math. Appl. Austral. Nat. Univ., Canberra, 2010, pp. 1–61.
- [36] J. VAN NEERVEN, M. C. VERAAR, AND L. WEIS, *Stochastic integration in UMD Banach spaces*, Ann. Probab., 35 (2007), pp. 1438–1478.
- [37] J. VAN NEERVEN AND L. WEIS, *Stochastic integration of functions with values in a Banach space*, Studia Math., 166 (2005), pp. 131–170.
- [38] B. WRÓBEL, *On g -functions for laguerre function expansions of Hermite type*, Proc. Indian Acad. Sci. (Math. Sci.), 121 (2011), pp. 45–75.
- [39] Q. XU, *Littlewood-Paley theory for functions with values in uniformly convex spaces*, J. Reine Angew. Math., 504 (1998), pp. 195–226.

JORGE J. BETANCOR, ALEJANDRO J. CASTRO, JUAN C. FARIÑA AND LOURDES RODRÍGUEZ-MESA

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA,

CAMPUS DE ANCHIETA, AVDA. ASTROFÍSICO FRANCISCO SÁNCHEZ, S/N,

38271, LA LAGUNA (STA. CRUZ DE TENERIFE), SPAIN

E-mail address: `jbetanco@ull.es`, `ajcastro@ull.es`, `jcfarina@ull.es`, `lrguez@ull.es`